

Research Article

Convergence Results for Common Fixed Points of Nearly Nonexpansive Mappings

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Received: 15.07.2026;**Accepted:** 15.09.2026;**Published:** 26.09.2026**Abstract**

This paper aims to establish several convergence results for common fixed points of two nearly nonexpansive mappings by means of a modified S-iteration process. Specifically, we prove both strong and weak convergence theorems in uniformly convex Banach spaces.



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1. Introduction

Fixed-point theory is a central and fast-growing area of nonlinear functional analysis that studies results ensuring the existence of fixed points, known as fixed-point theorems. These theorems are essential tools for proving both the existence and uniqueness of solutions in a wide range of mathematical models used to describe real-world phenomena. There is an extensive and continually growing body of research on fixed-point theory. Among its key results, the Banach contraction principle [1] stands out as a foundational theorem and is regarded as a central pillar of fixed-point theory. In 1965, Browder [2] introduced the concept of nonexpansive mapping and proved existence results for fixed points. It is well known that the Picard iteration [3] does not necessarily converge when applied to a nonexpansive mapping in a Banach space. Goebel and Kirk [4] generalized Browder's result to the broader class of asymptotically nonexpansive mappings. In 2005, Sahu [5] introduced the class of nearly Lipschitzian mappings, which significantly extends the class of Lipschitzian mappings. The study of fixed-point theory is largely driven by the approximation of fixed points in various settings for nonlinear mappings through different iterative procedures. By way of illustration, we highlight only a few of these: Mann iteration [6], Ishikawa iteration [7], Halpern iteration [8], Noor iteration [9], S-iteration [10]. Motivated by the preceding studies, we establish several convergence results for nearly nonexpansive mappings by employing a modified S-iteration scheme.

2. Preliminaries

Let C be a nonempty subset of a real Banach space X and $T: C \rightarrow C$ be a mapping with the fixed point set $F(T)$, i.e., $F(T) = \{p \in C : Tp = p\}$. Now, we recall some definitions and conclusions for our presentation.

Definition 2.1. ([4]) The function $\delta_X: [0, 2] \rightarrow [0, 1]$ is said to be the modulus of convexity of X if

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon \right\} \quad (1)$$

X is said to be uniformly convex if $\delta_X(0) = 0$ and $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Let $\mathcal{S} = \{x \in X : \|x\| = 1\}$. The space X has the following properties:

(i) *Gâteaux differentiable norm* if

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \quad (2)$$

exists for each x and y in S .

(ii) *Fréchet differentiable norm* [11] if for each x in S , the above limit exists and is attained uniformly for y in S and in this case, it is also well-known that

$$\begin{aligned} \langle h, J(x) \rangle + \frac{1}{2} \|x^2\| &\leq \frac{1}{2} \|x+h^2\| \\ &\leq \langle h, J(x) \rangle + \frac{1}{2} \|x^2\| + a(\|x\|) \end{aligned} \quad (3)$$

for all $x, h \in X$, where J is the Fréchet derivative of the functional $\frac{1}{2} \|\cdot\|^2$ at $x \in X$, $\langle \cdot, \cdot \rangle$ is the duality pairing between X and its dual X^* , and $a: [0, \infty) \rightarrow \mathbb{R}$ is an increasing function such that $\lim_{t \rightarrow 0} a(t)/t = 0$.

(iii) *Opial's condition* [12] if for any sequence $\{x_n\}$ in X , $x_n \rightharpoonup x$ it follows that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\| \quad (4)$$

for all $y \in X$ with $y \neq x$.

Definition 2.2. ([2]) A mapping $T: C \rightarrow C$ is said to be demiclosed at zero if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n \rightarrow 0$ then $Tx = 0$.

Definition 2.3. ([2]) A mapping $T: C \rightarrow C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C. \quad (5)$$

Definition 2.4. ([4]) A mapping $T: C \rightarrow C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad \text{for all } x, y \in C \text{ and } n \geq 1. \quad (6)$$

Definition 2.5. ([4]) A mapping $T: C \rightarrow C$ is said to be uniformly L -Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \quad \text{for all } x, y \in C \text{ and } n \geq 1. \quad (7)$$

Remark 2.6. It easy to see that every nonexpansive mapping T is asymptotically nonexpansive with sequence $k_n = 1$ and every asymptotically nonexpansive mapping is uniformly L -Lipschitzian with $L = \sup_{n \in \mathbb{N}} k_n$.

Definition 2.7. ([5]) Let $\{a_n\}$ be a sequence in $[0, 1]$ with $\lim_{n \rightarrow \infty} a_n = 0$. A mapping $T: C \rightarrow C$ is said to be nearly nonexpansive with respect to $\{a_n\}$ if

$$\|T^n x - T^n y\| \leq \|x - y\| + a_n \quad \text{for all } x, y \in C \text{ and } n \geq 1. \quad (8)$$

Now, it is clear that the nonexpansive mappings are asymptotically nonexpansive mappings, hence are nearly nonexpansive mappings.

Lemma 2.8 ([13]). Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \text{for all } n \in \mathbb{N}$$

If $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.9 ([14]). Let X be a real uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$, for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and $\lim_{n \rightarrow \infty} \|(1 - t_n)x_n + t_n y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

In 2007, Fukhar-ud-din and Khan [15] introduced the notion of mappings satisfying Condition (A) which is defined as follows:

Definition 2.10. [15] Two mappings $T_1, T_2: C \rightarrow C$ where C is a subset of a normed space X , are said to satisfy the Condition (A), if there exists a non-decreasing function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$ such that either $\|x - T_1x\| \geq \phi(d(x, F))$ or $\|x - T_2x\| \geq \phi(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

If $T_1 = T_2$ the Condition (A) reduces to the Condition of Senter and Dotson for approximating fixed points of nonexpansive mappings (see, [16]).

Now, we examine several well-known iterative schemes. Let C be a nonempty subset of normed space X and $T: C \rightarrow C$ a self-map. The Picard iteration scheme is defined by

$$x_{n+1} = Tx_n, \quad n \geq 0, \quad (9)$$

(see, for example [3]). Then the Mann iteration process [6] is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \geq 0, \quad (10)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ which satisfies the condition $\sum_{n=0}^{\infty} \alpha_n = \infty$. For any $u \in C$, Halpern [8] introduced the following iteration method

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \quad (11)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Also, the Ishikawa iteration process (see, for example [7]) is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \\ y_n = (1 - \beta_n)x_n + \beta_nTx_n \end{cases} \quad (12)$$

where $\{\alpha_n\}_{n \geq 0}$ and $\{\beta_n\}_{n \geq 0}$ are real sequences in $[0, 1]$ for all $n \geq 0$. The Noor iteration process [9] is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \\ y_n = (1 - \beta_n)x_n + \beta_nTz_n, \\ z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \end{cases} \quad n \geq 0, \quad (13)$$

where $\{\alpha_n\}_{n \geq 0}$, $\{\beta_n\}_{n \geq 0}$ and $\{\gamma_n\}_{n \geq 0}$ are real sequences in $[0, 1]$. Later, Agarwal et al. [10] introduced a modified S-iteration process for approximating fixed points of asymptotically nonexpansive mappings. This process is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \end{cases} \quad n \geq 0. \quad (14)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. We adapted the modified S-iteration process for two nonlinear mappings by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)T_1^n x_n + \alpha_n T_2^n y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T_1^n x_n, \end{cases} \quad n \geq 0. \quad (15)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and we establish several convergence results in the framework of uniformly convex Banach spaces.

3. Convergence Results

In this section we prove some strong and weak convergence theorems with respect to the iteration process (15), for nearly nonexpansive mappings in real uniformly convex Banach spaces. Throughout this paper, we assume that $a_n = \max\{a_n^{(1)}, a_n^{(2)}\}$, $L = \max\{L_1, L_2\}$ and $F = F(T_1) \cap F(T_2)$ is the set of common fixed points of the mappings T_1 and T_2 .

Lemma 3.1. Let C be a nonempty closed convex subset of a real Banach space X . Let $T_1, T_2: C \rightarrow C$ be two nearly nonexpansive mapping with respect to $\{a_n^{(1)}\}$ and $\{a_n^{(2)}\}$ such that $\sum_{n=0}^{\infty} a_n < \infty$. Let $\{x_n\}$ be the modified S-iteration defined by (15) and $F \neq \emptyset$. Then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T)$.

Proof. Let $p \in F$, by (15) we have

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)(x_n - p) + \beta_n(T_1^n x_n - p)\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n(\|x_n - p\| + a_n^{(1)}) \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n(\|x_n - p\| + a_n) \\ &\leq \|x_n - p\| + a_n. \end{aligned} \quad (16)$$

and

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)(T_1^n x_n - p) + \alpha_n(T_2^n y_n - p)\| \\ &\leq (1 - \alpha_n)(\|x_n - p\| + a_n^{(1)}) + \alpha_n(\|y_n - p\| + a_n^{(2)}) \\ &\leq (1 - \alpha_n)(\|x_n - p\| + a_n) + \alpha_n(\|y_n - p\| + a_n) \\ &\leq (1 - \alpha_n)(\|x_n - p\| + a_n) + \alpha_n(\|x_n - p\| + 2a_n) \\ &= \|x_n - p\| + (1 + \alpha_n)a_n. \end{aligned} \quad (17)$$

It follows from $\sum_{n=0}^{\infty} a_n < \infty$ and Lemma 3.1 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$. $\square$

Theorem 3.2. Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $T_1, T_2 : C \rightarrow C$ be two uniformly L_1, L_2 Lipschitzian and nearly nonexpansive mappings with respect to $\{a_n^{(1)}\}$ and $\{a_n^{(2)}\}$ such that $\sum_{n=0}^{\infty} a_n < \infty$. Let $\{x_n\}$ be defined by (15) then $\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0$.

Proof. By Lemma 3.1 $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$. Now, we set $\lim_{n \rightarrow \infty} \|x_n - p\| = r$.

From (16) we have

$$\|y_n - p\| \leq \|x_n - p\| + a_n.$$

which implies

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq r \quad (18)$$

Also,

$$\|T_1^n x_n - p\| \leq \|x_n - p\| + a_n^{(1)}$$

so, we have

$$\limsup_{n \rightarrow \infty} \|T_1^n x_n - p\| \leq r. \quad (19)$$

Similarly

$$\|T_2^n y_n - p\| \leq \|y_n - p\| + a_n^{(2)}$$

and we have

$$\limsup_{n \rightarrow \infty} \|T_2^n y_n - p\| \leq r. \quad (20)$$

On the other hand

$$r = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = \lim_{n \rightarrow \infty} \|(1 - \alpha_n)(T_1^n x_n - p) + \alpha_n(T_2^n y_n - p)\|.$$

Therefore by Lemma 3.5, we obtain

$$\lim_{n \rightarrow \infty} \|T_1^n x_n - T_2^n y_n\| = 0. \quad (21)$$

Now,

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)T_1^n x_n + \alpha_n T_2^n y_n - p\| \\ &\leq \|T_1^n x_n - p\| + \alpha_n \|T_1^n x_n - T_2^n y_n\| \end{aligned}$$

yields that

$$r \leq \liminf_{n \rightarrow \infty} \|T_1^n x_n - p\| \quad (22)$$

From (19) and (22) we have

$$\lim_{n \rightarrow \infty} \|T_1^n x_n - p\| = r. \quad (23)$$

From

$$\begin{aligned} \|T_1^n x_n - p\| &\leq \|T_1^n x_n - T_2^n y_n\| + \|T_2^n y_n - p\| \\ &\leq \|T_1^n x_n - T_2^n y_n\| + \|y_n - p\| + a_n^{(2)} \end{aligned}$$

we obtain

$$r \leq \liminf_{n \rightarrow \infty} \|y_n - p\|. \quad (24)$$

By using (18) and (24) we get

$$\lim_{n \rightarrow \infty} \|y_n - p\| = r. \quad (25)$$

Therefore

$$r = \lim_{n \rightarrow \infty} \|y_n - p\| = \lim_{n \rightarrow \infty} \|(1 - \beta_n)(x_n - p) + \beta_n(T_1^n x_n - p)\|.$$

gives by Lemma 3.5 that

$$\lim_{n \rightarrow \infty} \|x_n - T_1^n x_n\| = 0. \quad (26)$$

So, by $\|y_n - x_n\| = \beta_n \|T_1^n x_n - x_n\|$ and (26) we have

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (27)$$

Also,

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1 - \alpha_n)T_1^n x_n + \alpha_n T_2^n y_n - x_n\| \\ &\leq \|T_1^n x_n - x_n\| + \alpha_n \|T_1^n x_n - T_2^n y_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

implies that

$$\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \|y_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (28)$$

and from

$$\|x_{n+1} - T_2^n y_n\| \leq \|x_{n+1} - x_n\| + \|x_n - T_1^n x_n\| + \|T_1^n x_n - T_2^n y_n\|$$

we get

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_2^n y_n\| = 0. \quad (29)$$

Since $T_1, T_2: C \rightarrow C$ are uniformly Lipschitzian mappings, we have

$$\begin{aligned} \|x_{n+1} - T_1 x_{n+1}\| &\leq \|x_{n+1} - T_1^{n+1} x_{n+1}\| + \|T_1^{n+1} x_{n+1} - T_1^{n+1} x_n\| + \|T_1^{n+1} x_n - T_1 x_{n+1}\| \\ &\leq \|x_{n+1} - T_1^{n+1} x_{n+1}\| + L\|x_{n+1} - x_n\| + L\|T_1^n x_n - x_{n+1}\| \\ &= \|x_{n+1} - T_1^{n+1} x_{n+1}\| + L\|x_{n+1} - x_n\| + L\alpha_n \|T_1^n x_n - T_2^n y_n\| \rightarrow 0 \end{aligned}$$

and it follows

$$\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0. \quad (30)$$

Now, from

$$\begin{aligned} \|x_n - T_2^n x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_2^n y_n\| + \|T_2^n y_n - T_2^n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_2^n y_n\| + L\|y_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Thus

$$\begin{aligned} \|x_{n+1} - T_2 x_{n+1}\| &\leq \|x_{n+1} - T_2^{n+1} x_{n+1}\| + \|T_2^{n+1} x_{n+1} - T_2 x_{n+1}\| \\ &\leq \|x_{n+1} - T_2^{n+1} x_{n+1}\| + L\|T_2^n x_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - T_2^{n+1} x_{n+1}\| + L\|T_2^n x_{n+1} - T_2^n y_n\| + L\|T_2^n y_n - x_{n+1}\| \\ &\leq \|x_{n+1} - T_2^{n+1} x_{n+1}\| + L\|x_{n+1} - y_n\| + L\alpha_n + L\|T_2^n y_n - x_{n+1}\| \end{aligned}$$

it follows

$$\lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0. \quad (31)$$

□

Theorem 3.3. Assume that all the conditions of Theorem 3.2 are satisfied. Then the sequence $\{x_n\}$ generated by (15) converges strongly to a fixed point of F if and only if

$$\liminf_{n \rightarrow \infty} d(x_n, F) = 0,$$

where $d(x, F) = \inf\{d(x, p) : p \in F\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. From Lemma 3.1 we know that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$, so $\lim_{n \rightarrow \infty} d(x_n, F)$ exists for all $p \in F$. Thus by hypothesis

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0. \quad (32)$$

Now we prove that $\{x_n\}$ is a Cauchy sequence in C . Indeed, from (17), we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + (1 + \alpha_n)a_n$$

Denote $b_n := (1 + \alpha_n)a_n$ and for any $m, n \in \mathbb{N}, m > n \geq 1$, we have

$$\begin{aligned} \|x_m - p\| &\leq \|x_{m-1} - p\| + b_{m-1} \\ &\leq \|x_{m-2} - p\| + b_{m-1} + b_{m-2} \\ &\vdots \\ &\leq \|x_n - p\| + \sum_{i=n}^{m-1} b_i \\ &\leq \|x_n - p\| + \sum_{i=n}^{\infty} b_i. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ and $\sum_{i=n}^{\infty} b_i < \infty$ for any $\varepsilon > 0$ there exists an integer n_0 such that

$$d(x_n, F) < \frac{\varepsilon}{4}, \quad \sum_{i=n_0}^{\infty} b_i < \frac{\varepsilon}{4}.$$

Hence, there exists $\hat{p} \in F$ such that

$$\|x_{n_0} - \hat{p}\| < \frac{\varepsilon}{4}, \quad \sum_{i=n_0}^{\infty} b_i < \frac{\varepsilon}{4}.$$

So, for all $m, n \geq n_0$, we get from the above inequality that

$$\begin{aligned} \|x_m - x_n\| &\leq \|x_m - \hat{p}\| + \|x_n - \hat{p}\| \\ &\leq \|x_{n_0} - \hat{p}\| + \sum_{i=n_0}^{\infty} b_i + \|x_{n_0} - \hat{p}\| + \sum_{i=n_0}^{\infty} b_i \\ &= 2\left(\|x_{n_0} - \hat{p}\| + \sum_{i=n_0}^{\infty} b_i\right) \\ &< 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon. \end{aligned}$$

Thus, it follows that $\{x_n\}$ is a Cauchy sequence. Since C is a closed subset of Banach space X , the sequence $\{x_n\}$ converges strongly to some $p \in C$. Since F is a closed subset of C and $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ we have $p \in F$. Hence, the sequence $\{x_n\}$ converges strongly to a fixed point of F . This completes the proof. $\square$

Theorem 3.4. Assume that all the conditions of Theorem 3.2 are satisfied and let T_1, T_2 two mappings that satisfy the Condition (A). Then the sequence $\{x_n\}$ generated by (15) converges strongly to a fixed point of F .

Proof. We showed in Theorem 3.3, that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists for all $p \in F$. From Theorem 3.1 we have $\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0$. It follows from Condition (A) that

$$\lim_{n \rightarrow \infty} \phi(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0,$$

or

$$\lim_{n \rightarrow \infty} \phi(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0.$$

Thus

$$\lim_{n \rightarrow \infty} \phi(d(x_n, F)) = 0.$$

Since $\phi: [0, \infty) \rightarrow [0, \infty)$ is a non-decreasing function with $\phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$, we get

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0$$

Consequently, $\{x_n\}$ converges strongly to a fixed point of F . $\square$

Lemma 3.5. Under the assumption of Theorem 3.2, if X has a Fréchet differentiable norm, then for all $p_1, p_2 \in F$, the following limit exists

$$\lim_{n \rightarrow \infty} \langle x_n, J(p_1 - p_2) \rangle$$

where $\{x_n\}$ is the sequence defined by (15). In particular, $\langle l_1 - l_2, J(p_1 - p_2) \rangle = 0$ for all $p_1, p_2 \in F$ and $l_1, l_2 \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$.

Proof. Let $x = p_1 - p_2$ with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in inequality (3). Then, we have

$$t \langle x_n - p_1, J(p_1 - p_2) \rangle + \frac{1}{2} \|p_1 - p_2\|^2 \leq \frac{1}{2} \|tx_n + (1-t)p_1 - p_2\|^2$$

$$\frac{1}{2} \|tx_n + (1-t)p_1 - p_2\|^2 \leq t \langle x_n - p_1, J(p_1 - p_2) \rangle + \frac{1}{2} \|p_1 - p_2\|^2 + a(t\|x_n - p_1\|).$$

Since $\sup_{n \geq 1} \|x_n - p_1\| \leq M$ for some $M > 0$, we have

$$t \limsup_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle + \frac{1}{2} \|p_1 - p_2\|^2 \leq \frac{1}{2} \lim_{n \rightarrow \infty} \|tx_n + (1-t)p_1 - p_2\|^2$$

$$\frac{1}{2} \lim_{n \rightarrow \infty} \|tx_n + (1-t)p_1 - p_2\|^2 \leq t \liminf_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle + \frac{1}{2} \|p_1 - p_2\|^2 + a(tM).$$

Thus,

$$\limsup_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle \leq \liminf_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle + \frac{a(tM)}{tM} M.$$

Letting $t \rightarrow 0$, then $\lim_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle$ exists for all $p_1, p_2 \in F$; in particular, we have $\langle l_1 - l_2, J(p_1 - p_2) \rangle = 0$ for all $l_1, l_2 \in \omega_w(\{x_n\})$. $\square$

Theorem 3.6. Under the assumption of Theorem 3.2, if X has a Fréchet differentiable norm, then the sequence $\{x_n\}$ defined by (15) converges weakly to a common fixed point of the mappings T_1 and T_2 .

Proof. From Lemma 3.5, we get $\langle l_1 - l_2, J(p_1 - p_2) \rangle = 0$ for all $l_1, l_2 \in \omega_w(\{x_n\})$. Therefore $\|p - q\|^2 = \langle p - q, J(p - q) \rangle = 0$ implies $p = q$. Hence, $\{x_n\}$ converges weakly to a common fixed point in F . This completes the proof. $\square$

Theorem 3.7. Under the assumptions of Theorem 3.2, if X satisfies Opial's condition and the mappings $I - T_1$ and $I - T_2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by (15) converges weakly to a common fixed point of the mappings T_1 and T_2 .

Proof. Let $p \in F$, from Lemma 3.1 the sequence $\{\|x_n - p\|\}$ is bounded. Since X is uniformly convex, every bounded subset of X is weakly compact. Thus there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $\hat{p} \in C$. From Theorem 3.2, we have

$$\lim_{k \rightarrow \infty} \|x_{n_k} - T_1 x_{n_k}\| = 0 \text{ and } \lim_{k \rightarrow \infty} \|x_{n_k} - T_2 x_{n_k}\| = 0.$$

Since the mappings $I - T_1$ and $I - T_2$ are demiclosed at zero, therefore $T_1 \hat{p} = \hat{p}$ and $T_2 \hat{p} = \hat{p}$ for which means $\hat{p} \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to $\hat{p}$. Suppose on contrary that there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $\hat{q} \in C$ and $\hat{p} \neq \hat{q}$. Then, by applying the same method described above, we can also prove that $\hat{q} \in F$. From Lemma 3.1 the limits $\lim_{n \rightarrow \infty} \|x_n - \hat{p}\|$ and $\lim_{n \rightarrow \infty} \|x_n - \hat{q}\|$ exist. From the Opial condition of X , we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - \hat{p}\| &= \lim_{n_k \rightarrow \infty} \|x_{n_k} - \hat{p}\| \\ &< \lim_{n_k \rightarrow \infty} \|x_{n_k} - \hat{q}\| \\ &= \lim_{n \rightarrow \infty} \|x_n - \hat{q}\| \\ &= \lim_{n_j \rightarrow \infty} \|x_{n_j} - \hat{q}\| \\ &< \lim_{n_j \rightarrow \infty} \|x_{n_j} - \hat{p}\| \\ &= \lim_{n \rightarrow \infty} \|x_n - \hat{p}\|, \end{aligned}$$

which is a contradiction, so $\hat{p} = \hat{q}$. Hence, $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1 and T_2 . This completes the proof. $\square$

4. Concluding remarks

In this paper, we study the modified iteration scheme (15) for two nearly nonexpansive self mappings and establish some strong convergence and weak convergence theorems using the following conditions: (i) the space X has a Fréchet differentiable norm; (ii) the space X satisfies Opial's condition.

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