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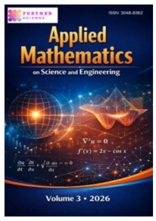

Research Article

Some remarks on G-nonexpansive and G-quasi nonexpansive mappings

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Abstract

Here, we discussed some relations between G-nonexpansive mappings and $G-$quasi-nonexpansive mappings and their structures. Further we obtained some convergence results of a sequence, defined by Schaefer iteration scheme for multi-valued Osilike-Berinde G-nonexpansive mapping in uniformly convex hyperbolic metric space. Also the results obtained in the paper are justified with some examples. To show the significance of results, an application is discussed here.

1. Introduction

Fixed point theory is an important tool for finding solutions of the problems in the form of equations or inequalities. Fixed point theory is a rich, interesting and highly applied branch of mathematics. There exists many classical fixed point theorems which inspire research activities around them.

The earliest fixed point theorem is due to L.E.J. Brouwer [1] which asserts that "A continuous mapping of a closed unit ball in $\mathbb{R}^n$ has at least one fixed point". From its origin, the Brouwer's theorem is considered as the source of inspiration for topologist and the existing literature has witnessed various generalizations, extensions, and alternative proofs of this classical theorem till date.

On the other hand, the first result in metric fixed point theory was given by Banach [2] which is popularly referred as 'Banach Contraction Principle' which stated that "Every contraction on a complete metric space always has a unique fixed point". This principle has been generalized in many directions.

Recall that if X is any non-empty set and $J : \mathrm{X} \to \mathrm{X}$ such that $u = Ju$, and if $J : \mathrm{X} \to P(\mathrm{X})$ such that $u \in Ju$, then u is called fixed point with respect to single-valued and multi-valued mapping J. Recall that a mapping $J : \mathrm{X} \to \mathrm{X}$ is called contraction, if there exists a $\lambda \in (0,1)$ such that

$$d(Ju, Jv) \leq \lambda d(u, v).$$

Contraction mappings are further generalized in many directions. In this continuity, Osilike [3] introduced the following mapping:

Definition 1.1. *Let* K *be a non-empty subset of a metric space* X*. A mapping* $J : \mathrm{K} \to \mathrm{K}$ *is called Osilike mapping if there exists a* $\lambda \in [0,1]$ *and* $L \in [0,\infty]$ *such that*

$$d(Ju,Jv) \leq \lambda d(u,v) + Ld(u,Ju), \ \forall u,v \in \mathrm{K}.$$

This concept was further extended by Berinde and Berinde [4] to a multi-valued mapping in the following manner:

Definition 1.2. *[4] A multi-valued mapping* $J : \mathrm{K} \to CB(\mathrm{K})$ *is called weak contraction if there exists a* $\lambda \in [0,1]$ *and* $L \in [0,\infty]$ *such that*

$$H(Ju,Jv) \leq \lambda d(u,v) + Ld(u,Jv), \ \forall \ u,v \in \mathrm{K}.$$

Remark 1.3. *In above definition, when* $\lambda = 1$*,* J *is called Osilike-Berinde nonexpansive mapping.*

In 2019, Bunlue and Suantai [5] proved the existence of fixed points as well as the demi-closed principle for Osilike-Berinde-nonexpansive mappings in Banach spaces satisfying the Opial's condition. The results in [5] was obtained for $CAT(0)$ space by Klangpraphan and Panyanak [6].

On the other hand, Jachymski [7] combined the concepts of fixed point theory and graph theory to prove a generalization of the Banach contraction principle in a complete metric space endowed with a graph. In 2010, Beg et al. [8] extended Jachymski's result to the general setting of a multi-valued $G-$contraction. Later on, Alfuraidan and Khamsi [9] introduced the notion of multi-valued $G-$nonexpansive mappings and proved the existence of fixed points for such kind of mappings in hyperbolic metric spaces.

In this paper, first section contains some properties of Osilike-Berinde $G-$nonexpansive mappings and $G-$quasi nonexpansive mappings and its structure in a complete hyperbolic space. Second section contains some fixed point results of a sequence defined by iteration scheme.

$$u_{k+1} = (1-\lambda)u_k + \lambda Ju_k, \ \lambda \in (0,1). \tag{1}$$

The iteration scheme (1) was introduced by Schaefer [10], where he extended the result of Krasnoselskii [11] by considering iteration scheme (1). In the iteration scheme (1), if $\lambda = \frac{1}{2}$, then it converted into Krasnoselskii iteration scheme. Following is the version of iteration scheme (1) in the hyperbolic space:

$$u_{k+1} = d(W(\lambda, u_k, Ju_k))$$

Throughout the paper, F_J denotes the set of fixed point of J. In this paper, some relations between nonexpansive mappings, $G-$nonexpansive mappings, quasi-nonexpansive and $G-$quasi nonexpansive mappings are obtained. Also some strong convergence results of a sequence generated by iteration scheme (1) are proved for multi-valued Osilike-Berinde $G-$nonexpansive mappings in the framework of complete hyperbolic space.

2. Preliminaries

This section starts with some useful notations and elementary concepts.

$$\mathbb{C}B(\mathrm{K}) = Collection \ of \ all \ non-empty \ closed \ bounded \ subsets \ of \ \mathrm{K},$$

and $H(.,.)$ be the Hausdorff distance on $\mathbb{C}B(\mathrm{K})$ which is defined by

$$H(A,B) = \max\{\sup d(u,B), \sup d(A,v)\},$$

where $A,B \in \mathbb{C}B(\mathrm{K})$, $d(A,u) = \inf_{a \in A} ||a-u||$.

Definition 2.1. *[12] A graph is an ordered pair* (V,E)*, where* V *is a set and* E *is a binary relation on* V*,* $E \subseteq V \times V$*. Elements of* E *are called edges. Given a graph* $G = (V,E)$*, a path of* G *is a sequence* $a_0a_1...a_{n-1}...$ *with* $(a_i,a_{i+1}) \in E$ *for each* $i = 0,1,2,...$*. A graph is connected if there is a finite path joining any two of its vertices.*

Definition 2.2. *[13] A directed graph also called a digraph is a graph in which the edges have a direction.*

Definition 2.3. *[7] A mapping* $J : \mathrm{X} \to \mathrm{X}$ *is called* $G-$ *continuous if for* $u \in \mathrm{X}$ *and* $\{u_k\}$ *be a sequence in* X *such that* $u_k \to u$ *and* $(u_k,u_{k+1}) \in E(G)$ *for* $k \in \mathbb{N}$*,* $Ju_k \to Ju$*.*

Definition 2.4. *[14] Let* $G = (V(G),E(G))$ *be a directed graph. A graph* G *is called transitive if for any* $u,v,w \in V(G)$ *such that* $(u,v) \in E(G)$ *and* $(v,w) \in E(G)$*, then* $(u,w) \in E(G)$*.*

Definition 2.5. *[15] A hyperbolic space* (X,d,W) *is a metric space* (X,d) *together with a convexity mapping* $W : \mathrm{X} \times \mathrm{X} \times [0,1] \to \mathrm{X}$ *such that* $\forall\ x,y,z \in \mathrm{X}$ *and* $\alpha,\beta \in [0,1]$*,*

(i) $d(x,W(u,v,\alpha)) \leq (1-\alpha)d(x,v) + \alpha d(x,u)$,
(ii) $d(W(u,v,\alpha),W(u,v,\beta)) = |\alpha-\beta|d(u,v)$,
(iii) $W(u,v,\alpha) = W(v,u,1-\alpha)$
(iv) $d(W(u,w,\alpha),W(v,t,\alpha)) \leq (1-\alpha)d(w,t) + \alpha d(u,v)$.

Example 2.6. *[16] Let* $\mathrm{X} = \mathbb{R}$ *and* $d : \mathrm{X} \times \mathrm{X} \to [0, \infty]$ *be a mapping defined by*

$$d(u, v) = ||u - v||.$$

It is clear that d *is metric on* X. *Let* $\mathrm{K} = [0, 1]$ *be a subset of* X. *Further define a mapping* $W : \mathrm{X} \times \mathrm{X} \times [0, 1]$ *by*

$$W(u, v, \alpha) = \alpha u + (1 - \alpha)v,$$

$\forall\, u, v \in \mathrm{X}$ *and* $\alpha \in [0, 1]$. *Then* (X, d, W) *is hyperbolic space.*

Definition 2.7. *[17] A non-empty subset* K *of a hyperbolic space* X *is said to be convex, if* $W(u, v, \alpha) \in \mathrm{K}$, $\forall\, u, v \in \mathrm{K}$ *and* $\alpha \in [0, 1]$.

Definition 2.8. *[18] A hyperbolic space* X *is said to be uniformly convex if for any* $r > 0$ *and* $\varepsilon \in (0, 2]$, $\exists$ *a* $\delta \in (0, 1]$ *such that* $\forall\, u, v, w \in \mathrm{X}$,

$$d(W(u, v, \frac{1}{2}), w) \leq (1 - \delta)r,$$

provided $d(u, w) \leq r$, $d(v, w) \leq r$ *and* $d(u, v) \geq \varepsilon r$.

Definition 2.9. *[19] A sequence* $\{u_k\}$ *in* X *is said to be* $\Delta-$ *converge to* $u \in \mathrm{X}$, *if* u *is a unique asymptotic center of* $\{u_{k_n}\}$ *of* $\{u_k\}$. *In this case* $\Delta - \lim\limits_{k \to \infty} u_k = u$.

Definition 2.10. *[18] Let* X *be a hyperbolic space. A map* $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ *which provides such a* $\delta = \eta(r, \varepsilon)$ *for a given* $r > 0$ *and* $\varepsilon \in (0, 2]$ *is known as a modulus of uniform convexity of* X. *The mapping* η *is said to be monotone, if it decreases with* r.

Definition 2.11. *[19] Let* K *be a non-empty closed subset of a complete metric space* X *and* $\{u_k\}$ *be a sequence in* K. *Then* $\{u_k\}$ *is called Fejer monotone sequence with respect to* K, *if* $\forall\ u \in \mathrm{K}$ *and* $k \in \mathbb{N}$,

$$d(u_{k+1}, x) \leq d(u_k, u).$$

Definition 2.12. *[9] Let* K *be a non-empty subset of a hyperbolic space* X. *Let* $G = (V(G), E(G))$ *be a directed graph such that* $V(G) = \mathrm{K}$ *and* $E(G)$ *contains all the loops, i.e.,* $(u, u) \in E(G)$ *for any* $u \in V(G)$. *A mapping* $J : \mathrm{K} \to \mathrm{K}$ *is called an edge-preserving mapping (or* $G-$ *edge preserving mapping) if*

$$\forall\ u, v \in \mathrm{K}, (u, v) \in E(G) \Rightarrow (J(u), J(v)) \in E(G).$$

Definition 2.13. *[20] Let* K *be a non-empty closed convex subset of a hyperbolic space* X, $G = (V(G), E(G))$ *be a directed graph such that* $V(G) = \mathrm{K}$. *Then* K *is said to have property* (G) *if* $\{u_k\}$ *is any sequence in* K *such that* $\Delta - \lim_{k \to \infty} u_k = u$, *there exists a subsequence* $\{u_{k_n}\}$ *of* $\{u_k\}$ *such that* $(u_{k_n}, u) \in E(G)$ *for all* n

Definition 2.14. *[14] Let* K *be a non-empty convex subset of a hyperbolic space* X, $G = (V(G), E(G))$ *be a graph such that* $V(G) = \mathrm{K}$ *and* $J : \mathrm{K} \to \mathrm{K}$. *Then* J *is said to be* $G-$ *nonexpansive if the following conditions hold:*

(i) J *is an edge-preserving;*
(ii) $d(Ju, Jv) \leq d(u, v)$, *whenever* $(u, v) \in E(G)$ *for any* $u, v \in \mathrm{K}$.

Proposition 2.15. *[21] Let* K *be a non-empty subset of a hyperbolic space* X. *Suppose that* $\{u_k\}$ *is Fejer monotone sequence with respect to* K. *Then the followings are hold:*

(a) Sequence $\{u_k\}$ *is bounded.*
(b) For every $u \in K$, $\{d(u_k, u)\}$ *converges.*

Definition 2.16. *[20] A multi-valued mapping* $J : \mathrm{K} \to CB(\mathrm{K})$ *is called Osilike-Berinde* $G-$*nonexpansive, if*

- J *is an edge-preserving;*
- J *Osilike-Berinde nonexpansive mapping.*

Example 2.17. *Consider* $\mathrm{X} = \mathbb{R}$ *with metric* d *defined by* $d(u, v) = |u - v|$ *and* $\mathrm{K} = [0, 1]$. *Define* $W : \mathrm{X} \times \mathrm{X} \times [0, 1] \to \mathrm{X}$ *by*

$$W(u, v, \eta) = \eta u + (1 - \eta)v,\ \ \forall\ u, v \in \mathrm{X},\ \ \eta \in [0, 1].$$

Then (X, d, W) *is complete hyperbolic metric space and* K *is non-empty compact subset of* X. *Now define a mapping* $J : \mathrm{K} \to CB(\mathrm{K})$ *by*

$$Ju = \begin{cases} [0, 1 - u\cos\frac{1}{u}], & u \in [0, \frac{1}{6}); \\ [0, \frac{u+5}{6}], & u \in [\frac{1}{6}, 1]. \end{cases}$$

To prove that J *is multi-valued Osilike-Beringe nonexpansive mapping. For this consider the following cases:*
Case (i): when $u, v \in [0, \frac{1}{6})$, *then*

$$\begin{aligned} H(Ju, Jv) &= |u\cos\frac{1}{u} - v\cos\frac{1}{v}| \\ &\leq |u| + |v| \\ &= d(u, v), \\ d(u, Ju) &= |u - (1 - u\cos\frac{1}{u})| \\ &\leq |u|.|1 + \cos\frac{1}{u}| + 1 \\ &\leq |u| + 3 \end{aligned}$$

Clearly $H(Ju,Jv) \leq d(u,v) + Ld(u,Ju)$.
Case (ii): when $u,v \in [\frac{1}{6},0]$, then $H(Ju,Jv) = \frac{1}{6}d(u,v)$ and

$$\begin{aligned} d(u,Ju) &= |u - \frac{u+5}{6}| \\ &= \frac{5}{6}|u-1|. \end{aligned}$$

Since

$$\begin{aligned} &\frac{1}{6}d(u,v) < d(u,v) \\ &\frac{1}{6}d(u,v) < d(u,v) + \frac{5}{6}|u-1|. \end{aligned}$$

Clearly $H(Ju,Jv) \leq d(u,v) + Ld(u,Ju)$.
Case (iii): when $u \in [0,\frac{1}{6})$ and $v \in [\frac{1}{6},0]$,

$$\begin{aligned} H(Tu,Tv) &= |1 - ucos\frac{1}{u} - \frac{v+5}{6}| \\ &\leq 1 + |u| + |\frac{v+5}{6}| \\ &\leq 2; \\ d(u,Tu) &= |u - (1 - u\cos\frac{1}{u})| \\ &\leq |u|.|1 + \cos\frac{1}{u}| + 1 \\ &\leq |u| + 3 \\ &< 4. \end{aligned}$$

Clearly $H(Ju,Jv) \leq d(u,v) + Ld(u,Ju)$.
Similar case can be obtained by considering $v \in [0,\frac{1}{6})$ and $u \in [\frac{1}{6},0]$.
Further if X is endowed with a directed graph G such that $V(G) = \mathrm{K}$, $u,v \in V(G)$ with $|u| = |v| = 1$ and $(u,v) \in E(G)$ if $d(u,v) \leq 1$, Then J is multi-valued Osilike-Berinde nonexpansive mapping.

Definition 2.18. *[20] J is called $G-$quasi nonexpansive if $F_J \neq \emptyset$ and*

- *J is an edge-preserving;*
- *$H(Ju,p) \leq d(u,p)$, $\forall\, u \in \mathrm{K}$ and $p \in F_J$.*

Definition 2.19. *[19] Let K be a non-empty subset of a metric space X and $\{u_k\}$ be any bounded sequence in K. For $u \in \mathrm{X}$, there is a continuous functional $r(.,\{u_k\}) : \mathrm{X} \to [0,\infty)$ defined by*

$$r(u,\{u_k\}) = \limsup_{k\to\infty} d(u_k,u).$$

The asymptotic radius $r(\mathrm{K},\{u_k\})$ of $\{u_k\}$ with respect to K is given by

$$r(\mathrm{K},\{u_k\}) = \inf\{r(u,\{u_k\}) : u \in \mathrm{K}\}.$$

A point $u \in \mathrm{K}$ is said to be an asymptotic center of the sequence $\{u_k\}$ with respect to K, if

$$r(u,\{u_k\}) = \inf\{r(v,\{u_k\}) : v \in \mathrm{K}\}.$$

The set of all asymptotic centers of $\{u_k\}$ with respect to K is denoted by $A(\mathrm{K},\{u_k\})$.

Vetro [22] established some results related to Hausdorff distance. These results are as follows:

Lemma 2.20. *[22] Let (X,d) be a metric space. For any $A,B,C \in \mathbb{C}B(\mathrm{X})$ and any $u,v \in \mathrm{X}$, we have the following*

(i) $d(u,B) \leq d(u,b)$ *for* $b \in B$;
(ii) $\delta(A,B) \leq H(A,B)$;
(iii) $d(u,B) \leq H(A,B)$, *for any* $u \in A$;
(iv) $H(A,A) = 0$;
(v) $H(A,B) = H(B,A)$;
(vi) $H(A,C) \leq H(A,B) + H(B,C)$;
(vii) $d(u,A) \leq d(u,v) + d(v,A)$

Lemma 2.21. *[23] Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequences $\{u_k\}$ in X has a unique asymptotic center with respect to any non-empty closed convex subset K of X.*

3. Main Results

Theorem 3.1. *Let* K *be a non-empty subset of a hyperbolic space* X *and* $J : \mathrm{K} \to CB(\mathrm{K})$ *be a multi-valued mapping. Suppose that* X *is endowed with a directed graph with* $V(G) = \mathrm{K}$ *and* $E(G)$ *contains all the loops. Then*

(i) J is Osiloke-Berinde nonexpansive $\iff$ J is Osiloke-Berinde $G-$nonexpansive.
(ii) J is quasi-nonexpansive $\iff$ J is $G-$quasi-nonexpansive.

Proof.

(i) Suppose that J is Osiloke-Berinde nonexpansive, then by definition

$$H(Ju,Jv) \leq d(u,v) + Ld(u,Ju), \ \forall u,v \in \mathrm{K}.$$

Since $E(G) = \mathrm{K} \times \mathrm{K}$, hence for $u,v \in \mathrm{K}$, $(u,v) \in E(G)$, i.e. J is an edge-preserving. This shows that J is Osiloke-Berinde $G-$nonexpansive. Conversely, suppose that J is Osiloke-Berinde $G-$nonexpansive. Then it is Osiloke-Berinde nonexpansive.

(ii) Suppose that J is J is quasi-nonexpansive, then

$$H(Ju,Jv) \leq d(u,v).$$

Since $E(G) = \mathrm{K} \times \mathrm{K}$, hence for $u,v \in \mathrm{K}$, $(u,v) \in E(G)$hence for $u,v \in \mathrm{K}$, $(u,v) \in E(G)$, i.e. J is an edge-preserving. This shows that J is $G-$quasi-nonexpansive.
conversely, suppose that J is $G-$quasi-nonexpansive, then it is quasi-nonexpansive. □

Example 3.2. *Consider* $\mathrm{X} = \mathbb{R}$ *with metric d defined by* $d(u,v) = |u - v|$ *and* $\mathrm{K} = [0,1]$*. Define* $W : \mathrm{X} \times \mathrm{X} \times [0,1] \to \mathrm{X}$ *by*

$$W(u,v,\eta) = \eta u + (1-\eta)v, \ \forall \ u,v \in \mathrm{X}, \ \eta \in [0,1].$$

Then (X,d,W) *is complete hyperbolic metric space and* K *is non-empty compact subset of* X*. Assign a directed graph G by* $V(G) = \mathrm{K}$ *and* $E(G) = \mathrm{K} \times \mathrm{K}$*. Now define a mapping* $J : \mathrm{K} \to CB(\mathrm{K})$ *by*

$$Ju = \begin{cases} [0, \frac{u}{u+1} \cos \frac{1}{u}], u \neq 0; \\ \{0\}, u = 0. \end{cases}$$

Clearly $F_J = \{0\}$*. For* $u \in [0,1)$

$$\begin{aligned} H(Ju,J0) &= |\frac{u}{u+1} \cos \frac{1}{u}| \\ &\leq |\frac{u}{u+1}| \\ &< d(u,0). \end{aligned}$$

This implies that J is multi-valued quasi-nonexpansive mapping. Now for $k \in \mathbb{N}$, choose $v_k = \frac{1}{2k\pi}$ and $u_k = \frac{1}{2k\pi + \frac{\pi}{2}}$, then

$$\begin{aligned} H(Ju_k,Jv_k) &= |\frac{u_k}{u_k+1} \cos \frac{1}{u_k} - \frac{v_k}{v_k+1} \cos \frac{1}{v_k}| \\ &= \frac{1}{2\pi k + 1}, \\ d(u_k,v_k) &= \frac{\pi|}{2k(4\pi k + \pi)}, \\ d(u_k,Tu_k) &= \frac{1}{2k\pi + \frac{\pi}{2}} \\ &= \frac{2}{4k\pi + \pi}, \\ \frac{H(Ju_k,Jv_k) - d(u_k,v_k)}{d(u_k,Ju_k)} &= \frac{\frac{1}{2\pi k+1} - \frac{\pi}{2k(4\pi k+\pi)}}{\frac{2}{4k\pi+\pi}} \\ &\to 0. \end{aligned}$$

Hence J is not Osilike-Berinde nonexpansive mapping. From Theorem 3.1, J is not Osilike-Berinde $G-$nonexpansive mapping.

Theorem 3.3. *Let* K *be a non-empty subset of a hyperbolic space* X *and* $J : \mathrm{K} \to CB(\mathrm{K})$ *be a multi-valued mapping. If T is Osiloke-Berinde nonexpansive and* $F_T \neq \emptyset$*, then J is quasi-nonexpansive.*

Proof. Suppose that $w \in F_J$ and

$$\begin{aligned} H(Ju,Jw) &\leq d(u,w) + Ld(w,Jw) \\ &\leq d(u,w). \end{aligned}$$

Hence J is quasi-nonexpansive. □

Theorem 3.4. *Let* K *be a non-empty subset of a hyperbolic space* X *and* $J : \mathrm{K} \to CB(\mathrm{K})$ *be a multi-valued mapping. If* J *is Osiloke-Berinde* $G-$*nonexpansive and* $F_J \neq \emptyset$*, then* J *is* $G-$*quasi-nonexpansive.*

Proof. Suppose that $w \in F_J$ and

$$\begin{aligned} H(Ju, Jw) &\leq d(u, w) + Ld(w, Jw) \\ &\leq d(u, w). \end{aligned}$$

Hence J is $G-$quasi-nonexpansive. □

Theorem 3.5. *Let* K *be a non-empty subset of a hyperbolic space* X *and* $J : \mathrm{K} \to CB(\mathrm{K})$ *be a multi-valued Osiloke-Berinde* $G-$*nonexpansive mapping. Suppose that* X *is endowed with a directed graph with* $V(G) = \mathrm{K}$ *and* $E(G)$ *contains all the loops. Then* $I - J$ *is demiclosed.*

Proof. Let $\{u_k\}$ be a sequence in in K such that $\Delta - \lim u_k = u$ and $\lim_{k\to\infty} d(u_k, Ju_k) = 0$. Let $v_k \in Ju_k$ and $w_k \in Ju$ such that $d(u_k, v_k) = d(u_k, Ju_k), d(v_k, w_k) = d(v_k, Ju)$. Since K has property (G), there exists a subsequence $\{u_{k_n}\}$ such that $(u_{k_n}, u) \in E(G)$. Since Ju is compact, $\lim_{k\to\infty} w_{k_n} = w \in Ju$. Now J is Osiloke-Berinde $G-$nonexpansive mapping,

$$\begin{aligned} d(u_{k_n}, w) &\leq d(u_{k_n, v_{k_n}}) + d(y_{k_n}, w_{k_n}) + d(w_{k_n}, w) \\ &\leq d(u_{k_n, v_{k_n}}) + H(Ju_{k_n}, Ju) + d(w_{k_n}, w) \\ &\leq (1+L) d(u_{k_n}, Ju_{k_n}) + d(u_{k_n}, u) + d(w_{k_n}, w) \\ &\Rightarrow \lim_{k\to\infty} \sup d(u_{k_n}, u) \\ &\leq \lim_{k\to\infty} \sup d(w_{k_n}, u). \end{aligned}$$

This implies that $w \in A(\mathrm{K}, \{u_{k_n}\}) = \{u\}$. Hence $u = w \in Ju$. □

Lemma 3.6. *Let* K *be a non-empty closed convex subset of a complete hyperbolic space* X*. Let* $J : \mathrm{K} \to \mathbb{C}B(\mathrm{K})$ *be a multi-valued Osilike-Berinde* $G-$ *nonexpansive mapping such that* $F_J \neq \emptyset$ *with* $p \in F_J$*. Then* F_J *is closed and convex.*

Proof. First we show that F_J is closed. Let $\{u_k\}$ be a sequence in F_J such that $\{u_k\}$ converges to some $u \in \mathrm{K}$. Let $p \in F_J$, By using quasi-nonexpansiveness of J and Lemma 2.20, we have

$$\begin{aligned} d(u_k, Ju) &\leq d(u_k, p) + d(p, Ju) \\ &\leq H(Ju_k, Jp) + d(p, Ju) \\ &\leq d(u_k, p) + Ld(p, Jp) + d(p, Ju) \\ &\leq d(u_k, p) + d(p, u) \\ &\leq d(u_k, u). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have

$$\lim_{k\to\infty} d(u_k, Ju) = 0.$$

By uniqueness of limit, we have $u \in Tu$. Hence F_J is closed.

Next, to prove that F_J is convex. Let $u, v \in F_J$ and $\alpha \in [0,1]$. By using Lemma 2.20, we have

$$\begin{aligned} d(u, J(W(u, v, \alpha))) &\leq H(Ju, J(W(u, v, \alpha)) \\ &\leq d(u, W(u, v, \alpha)) + Ld(u, Ju) \\ &\leq d(u, W(u, v, \alpha)). \end{aligned}$$

Hence

$$d(u, J(W(u, v, \alpha))) \leq d(u, W(u, v, \alpha)) \tag{2}$$

Using similar argument, we have

$$d(v, J(W(u, v, \alpha))) \leq d(v, W(u, v, \alpha)). \tag{3}$$

By using (2), (3), and Lemma 2.20, we have

$$\begin{aligned} d(u, v) &\leq d(u, J(W(u, v, \alpha))) + d(J(W(u, v, \alpha)), v) \\ &\leq H(Ju, JW(u, v, \alpha)) + H(J(W(u, v, \alpha), Jv) \\ &\leq (d(u, W(u, v, \alpha)) + d(v, W(u, v, \alpha))) + L(d(u, Ju) + d(v, Jv)) \\ &\leq d(u, W(u, v, \alpha)) + d(v, W(u, v, \alpha)) \\ &= d(u, v). \end{aligned}$$

Therefore

$$d(u, v) \leq d(u, v). \tag{4}$$

Hence, (2) and (3) are $d(u,J(W(u,v,\alpha))) = d(u,W(u,v,\alpha))$ and
$d(v,J(W(u,v,\alpha))) = d(v,W(u,v,\alpha))$, because if strictly less than sign $<$ is considered, then from (4), $d(u,v) < d(u,v)$ which is a contradiction. Therefore

$$T(W(u,v,\alpha)) = W(u,v,\alpha),$$

$\forall\ u,v \in F_J$ and $\alpha \in [0,1]$. Thus $W(u,v,\alpha) \in F_J$, which implies that F_J is convex. □

Corollary 3.7. *Let* K *be a non-empty closed convex subset of a complete hyperbolic space* X*. Let* $T : \mathrm{K} \to \mathbb{C}B(\mathrm{K})$ *be multi-valued Osilike-Berinde* $G-$*nonexpansive mappings such that* $F_J =\neq \emptyset$ *with* $p \in F_J$*. Let* $\{u_k\}$ *be a bounded sequence in* K *such that* $\lim_{k\to\infty} d(u_k,Ju_k) = 0$*. Then* F_J *is closed and convex.*

Proof. Let $\{u_k\}$ be a bounded sequence in F_T such that $\{u_k\}$ converges to some $u \in \mathrm{K}$. Using quasi-nonexpansiveness of J

$$\begin{aligned} d(u_k,Jv) &\leq d(u_k,Ju_k) + d(Ju_k,p) + d(p,Jv) \\ &\leq d(u_k,Ju_k) + d(u_k,p) + d(p,v) \\ &\leq d(u_k,Ju_k) + d(u_k,v). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides,

$$\lim_{k\to\infty} d(u_k,Jv) = 0.$$

Hence F_J is closed. Rest of the proof is similar as in Lemma 3.6. □

Theorem 3.8. *Let* K *be a non-empty closed convex subset of a complete uniformly convex hyperbolic space* X *with monotone modulus of convexity* η*. Let* $J : \mathrm{K} \to \mathbb{C}B(\mathrm{K})$ *be a multi-valued Osilike-Berinde* $G-$*nonexpansive mapping such that* $F_J =\neq \emptyset$ *with* $p \in F_J$*. Let* $\{u_k\}$ *be a bounded sequence in* K *such that* $\lim_{k\to\infty} d(u_k,Jx_k) = 0$ *and* $\Delta - \lim_{k\to\infty} u_k = u^*$*. Then* $u^* \in F_J$*.*

Proof. Since $\{u_k\}$ is a bounded sequence in K, then from Lemma 2.21, $\{u_k\}$ has a unique asymptotic center in K. Since $\Delta - \lim_{k\to\infty} u_k = u^*$, we have $A(\{u_k\}) = \{u^*\}$. Observe that

$$\begin{aligned} d(u_k,Ju^*) &\leq d(u_k,Ju_k) + d(Ju_k,Ju^*) \\ &\leq d(u_k,Ju_k) + d(Ju_k,p) + d(p,Ju^*) \\ &\leq d(u_k,Ju_k) + d(u_k,p) + d(p,u^*). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have

$$\lim_{k\to\infty} d(u_k,Ju^*) \leq \lim_{k\to\infty} d(u_k,u^*).$$

Since

$$\begin{aligned} r(Ju^*,\{u_k\}) &= \limsup_{k\to\infty} d(u_k,Ju^*) \\ &\leq \limsup_{k\to\infty} d(u_k,u^*) \\ &= r(u^*,\{u_k\}). \end{aligned}$$

By uniqueness of asymptotic center of $\{u_k\}$, we have $Ju^* = u^*$. Hence $u^* \in F_J$. □

Theorem 3.9. *Let* K *be a non-empty closed convex subset of a complete uniformly convex hyperbolic space* X*. Let* $T : \mathrm{K} \to \mathbb{C}B(\mathrm{K})$ *be a multi-valued Osilike-Berinde* $G-$*nonexpansive mapping such that* $F_J =\neq \emptyset$ *with* $p \in F_J$*. Let* $\{u_k\}$ *be a bounded sequence in* K *, then* $\{u_k\}$ *converges to* $p \in F_J$ *iff* $\lim_{k\to\infty} d(u_k,Ju_k) = 0$*.*

Proof. Suppose that $\{u_k\}$ be a bounded sequence in K such that $\{u_k\}$ converges to $p \in F_J$, then $\lim_{k\to\infty} d(u_k,p)$ exists, hence $\lim_{k\to\infty} d(u_k,F_J) = 0$ for all $p \in F_J$.
If $\lim_{k\to\infty} d(u_k,F_J) = 0$, consequently, $\{u_k\}$ converges to $p \in F_J$. □

Example 3.10. *Consider* $\mathrm{X} = \mathbb{R}$ *with metric* d *defined by* $d(u,v) = |u-v|$ *and* $K = [0,3]$*. Define* $W : \mathrm{X}\times\mathrm{X}\times[0,1] \to \mathrm{X}$ *by*

$$W(u,v,\eta) = \eta u + (1-\eta)v,\ \ \forall\ u,v \in \mathrm{X},\ \ \eta \in [0,1].$$

Then (X,d,W) *is complete hyperbolic metric space and* K *is non-empty compact subset of* X*. Suppose that* X *is endowed with a directed graph* G *such that* $V(G) = \mathrm{K}$*,* $u,v \in V(G)$ *with* $|u| = |v| = 1$ *and* $(u,v) \in E(G)$ *if* $d(u,v) \leq 3$*. Now define a mapping* $J : \mathrm{K} \to CB(K)$ *by*

$$Ju = \begin{cases} [0,\frac{u}{3}], & u \neq 3; \\ \{0\}, & u = 3 \end{cases}$$

When $u,v = 3$*, then* $H(Tu,Tv) = 0$*, therefore* $(Ju,Jv) \in E(G)$*. Similarly, when* $u,v \neq 3$*, then* $H(Ju,Jv) = \frac{1}{3}d(u,v) < d(u,v)$*. Hence* $(Ju,Jv) \in E(G)$*. It shows that* J *is an edge-preserving multi-valued mapping.*
Next, to show that J *is Osilike-Berinde nonexpansive mapping, for this consider the following cases:*

Case (i): for $u,v=3$, $H(Ju,Jv)=0$ *and* $d(u,v)=0$ *and* $d(u,Ju)=3$. *Clearly* $H(Jx,Jy)\leq d(u,v)+Ld(u,Ju)$.
Case (ii): for $u,v\neq 3$, $H(Ju,Jv)=\frac{1}{3}d(u,v)$ *and*

$$\begin{aligned}
d(u,Ju) &= d(u,[0,\frac{u}{3}])\\
&= |u-\frac{u}{3}|\\
&= \frac{2}{3}|u|\\
&= \frac{2}{3}
\end{aligned}$$

Clearly $H(Ju,Jv)\leq d(u,v)+Ld(u,Ju)$.
Let $\{u_k=\frac{1}{2k}\}$ *be a bounded sequence in K and* $d(u_k,Ju_k)=\frac{1}{4k}\to 0$ *as* $k\to\infty$. *Clearly* $0\in F_J$. *Hence from the Theorem 3.9,* $\{u_k\}$ *converges to* $0\in F_J$.

4. Strong convergence results

This section contains convergence results of a sequence generated by Schaefer [10] iteration scheme (1).

Theorem 4.1. *Let* K *be a non-empty closed convex subset of a complete uniformly convex hyperbolic space* X *endowed with a directed graph G. Let* $J:\mathrm{K}\to\mathbb{C}B(\mathrm{K})$ *be a multi-valued Osilike-Berinde* $G-$*nonexpansive mapping such that* $F_J=\neq\emptyset$ *with* $p\in F_J$. *Let* $\{u_k\}$ *be a sequence in* K *defined by (1),* $d(u_k,p)$ *exists* $\forall\ p\in F_J$.

Proof. Since $F_J=\neq\emptyset$, so suppose that $p\in F_J$. Using Lemma 2.20

$$\begin{aligned}
d(u_k+1,p) &= d(W(\lambda,u_k,Ju_k),p)\\
&\leq (1-\lambda)d(u_k,p)+\lambda d(Ju_k,p)\\
&\leq (1-\lambda)d(u_k,p)+H(Ju_k,Jp)\\
&\leq (1-\lambda)d(u_k,p)+\lambda(d(u_k,p)+Ld(p,Jp))\\
&\leq \lambda d(u_k,p)\\
&\leq d(u_k,p).
\end{aligned}$$

This shows that $\{u_k\}$ is Fejer monotone with respect to $p\in F_J$. Hence from Proposition 2.15, $\{u_k\}$ is bounded sequence and $d(u_k,p)$ exists for all $p\in F_J$. □

Theorem 4.2. *Let* K *be a non-empty closed convex subset of a complete uniformly convex hyperbolic space* X *with monotone modulus of convexity* η *and endowed with a directed graph* G . *Let* $J:\mathrm{K}\to\mathbb{C}B(\mathrm{K})$ *be a multi-valued Osilike-Berinde* $G-$*nonexpansive mapping such that* $F_J=\neq\emptyset$ *with* $p\in F_J$. *Let* $\{u_k\}$ *be a sequence in* K *defined by (1), then* $\lim_{k\to\infty}d(u_k,Ju_k)=0$.

Proof. From Theorem 4.1, $d(u_k,p)$ exists $\forall\ p\in F_J$. Now

$$\begin{aligned}
d(u_k,Jx_k) &\leq d(u_k,p)+d(p,Ju_k)\\
&\leq d(u_k,p)+H(Jp,Ju_k)\\
&\leq d(u_k,p)+d(u_k,p)+Ld(p,Jp)\\
&= 2d(u_k,p)\to 0\ \ as\ \ k\to\infty.
\end{aligned}$$

Hence $\lim_{k\to\infty}d(u_k,Jx_k)=0$. □

Theorem 4.3. *Let* K *be a non-empty compact convex subset of a complete uniformly convex hyperbolic space* X *endowed with a directed graph G. Let* $J:\mathrm{K}\to\mathbb{C}B(\mathrm{K})$ *be a multi-valued Osilike-Berinde* $G-$*nonexpansive mapping such that* $F_J=\neq\emptyset$ *with* $p\in F_J$. *Let* $\{u_k\}$ *be a sequence in* K *defined by (1), then* $\{u_k\}$ *converges strongly to* $p\in F_J$ *iff* $\lim_{k\to\infty}d(u_k,F_J)=0$.

Proof. Let $\{u_k\}$ converges strongly to $p\in F_J$, then $\lim_{k\to\infty}d(u_k,p)=0$.
Since $d(u_k,F_J)=\inf\{d(u_k,p):p\in F_J\}$, hence $\lim_{k\to\infty}d(u_k,F_J)=0$.
Conversly, suppose that $\lim_{k\to\infty}d(u_k,F_J)=0$. Since

$$\begin{aligned}
&d(u_{k+1},p)\leq d(u_k,p)\\
&\qquad\Rightarrow d(u_{k+1},F_J)\leq d(u_k,F_J).
\end{aligned}$$

Hence from Proposition 2.15, $\lim_{k\to\infty}d(u_k,F_J)$ exists for $p\in F_J$. Next to prove $\{u_k\}$ is Cauchy sequence in K, form $k>n$,

$$\begin{aligned}
d(u_k,u_n) &\leq d(u_k,p)+d(p,u_n)\\
&= 2d(u_k,p)\\
&\leq 2\inf d(u_k,F_J)\\
&\to 0\ \ as\ \ k\to\infty.
\end{aligned}$$

This implies that $\{u_k\}$ is Cauchy sequence in K, therefore there exists $q \in K$ such that $u_k \to q$ as $k \to \infty$. Now

$$\begin{aligned} d(Jq,q) &\leq d(Jq,p)+d(p,q) \\ &\leq H(Jq,Jp)+d(p,q) \\ &\leq d(p,q)+Ld(p,Jp)+d(p,q) \\ &= 2d(p,q) \\ &\leq 2(d(p,u_k)+d(u_k,q)) \\ &\to 0 \ as \ k\to\infty. \end{aligned}$$

Hence $q \in F_J$. □

5. Application

In this section, the application of fixed point theory is shown in existence and uniqueness solution of a cantilever beam problem.

Theorem 5.1. *Consider a cantilever beam problem of fourth order*

$$\frac{d^4u}{dt^4} = M(t,u,\frac{du}{dt},\frac{d^2u}{dt^2},\frac{d^3u}{dt^3}), \ t\in[0,1] \tag{5}$$

with boundary condition

$$u(0) = \frac{du}{dt}(0) = \frac{d^2u}{dt^2}(1) = \frac{d^3u}{dt^3})(1) = 0,$$

where $M:\mathbb{I}\times\mathbb{R}^4 \to \mathbb{I}$ *is continuous function.*
Assume that for any $r>0$*,*

$$d(M(t,u,\frac{du}{dt}),M(t,v,\frac{dv}{dt}) \leq (1+e^{-r}-\frac{1}{\eta}),$$

then Equation (5) has a unique solution.

Proof. Suppose that $\mathrm{X} = C([0,1],\mathbb{R})$ and $W:\mathrm{X}\times\mathrm{X}\times[0,1]\to\mathrm{X}$ is defined by

$$W(u,v,\eta) = \eta u+(1-\eta)v, \ \forall \ u,v\in\mathrm{X}, \ \eta\in[0,1],$$

and $d(u,v) = \sup_{t\in[0,1]}|u(t)-v(t)|$. Then (X,d,W) is complete hyperbolic metric space. Now the Equation (5) can be written as

$$u(t) = \int_0^1 G(t,s)M(s,u(s),\frac{du}{dt}(s))ds,$$

where the Green function is given by

$$G(t,s) = \begin{cases} \frac{1}{6}t^2(3s-t), \ 0\leq t\leq s\leq 1; \\ \frac{1}{6}s^2(3t-s), \ 0\leq s\leq t\leq 1. \end{cases}$$

Now define a mapping $J:\mathrm{X}\to\mathrm{X}$ by

$$Ju(t) = \int_0^1 G(t,s)M(s,u(s),\frac{du}{dt}(s))ds,$$

then

$$\begin{aligned} d(Ju,Jv) &= d(Ju(t),Jv(t)) \\ &= \sup|Ju(t)-Jv(t)| \\ &\leq \sup|(1-\eta)(u(t)-v(t))+\eta(Tu(t)-Jv(t))| \\ &\leq \sup(1-\eta)|u(t)-v(t)|+\eta|Ju(t)-Jv(t)| \\ &\leq \sup(1-\eta)|u(t)-v(t)|+\eta|\int_0^1 G(t,s)M(s,u(s),\frac{du}{dt}(s))ds - \int_0^1 G(t,s)M(s,v(s),\frac{dv}{dt}(s))ds| \\ &\leq \sup(1-\eta)|u(t)-v(t)|+\eta|G(t,s)\int_0^1 (M(s,u(s),\frac{du}{dt}(s))-M(s,v(s),\frac{dv}{dt}(s)))ds| \\ &\leq \sup(1-\eta)|u(t)-v(t)|+\eta(1+e^{-r}-\frac{1}{\eta})|u(t)-v(t)||\int_0^1 G(t,s)ds| \\ &\leq \sup(1-\eta)|u(t)-v(t)|+\eta(1+e^{-r}-\frac{1}{\eta})|u(t)-v(t)||\int_0^t \frac{1}{6}s^2(3s-t)ds+\int_t^1 \frac{1}{6}t^2(3s-t)ds| \\ &\leq \sup(1-\eta)|u(t)-v(t)|+\eta(1+e^{-r}-\frac{1}{\eta})|u(t)-v(t)|\frac{1}{8}t^4+\frac{1}{12}(3t^2-2t^3-t^4)| \\ &\leq \frac{5}{8}e^{-r}d(u(t),v(t)) \\ &< e^{-r}d(u(t),v(t)). \end{aligned}$$

This implies that $d(Ju,Jv)\leq d(u,v)$. Interchanging the role of u and v, $H(Ju,Jv)\leq d(u,v)$. Since $d(u,v)\leq d(u,v)+Ld(u,Ju)$, $H(Ju,Jv)\leq d(u,v)+Ld(u,Ju)$. This shows that J is multi-valued Osilike-Berinde nonexpansive mapping. Form Theorem 3.1, J is Osilike-Berinde $G-$nonexpansive mapping. Further, if $F_J =\neq \emptyset$ with $p\in F_J$ and $\{u_k\}$ be a sequence in X defined by (1), then from Theorem 4.1, $d(u_k,p)$ exists $\forall \ p\in F_J$. □

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