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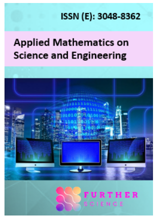

Research Article

An Inventory Model for Non-Instantaneously Reasoned, Continuously Deteriorating Items under a Linear Demand Rate

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Abstract

This paper develops a deterministic economic order quantity (EOQ) model for a single deteriorating item under a linear, time-dependent demand rate $D(t) = a + bt$, where deterioration proceeds at a constant instantaneous rate θ applied to the on-hand inventory. Shortages are not permitted. The governing first-order linear differential equation describing inventory depletion is solved in closed form, and the order quantity, holding cost integral, and deterioration cost expression are derived analytically and verified symbolically. The average total cost per unit time, $TCU(T)$, is shown via a rigorous second-order condition together with numerical evidence to be strictly convex in the cycle length T, guaranteeing a unique global minimizer T^*. The model is further validated by showing that it reduces exactly to the classical no-deterioration linear-trend model as $\theta \to 0$, to the constant-demand Ghare–Schrader deteriorating inventory model as $b \to 0$, and to the textbook economic order quantity formula as both $\theta \to 0$ and $b \to 0$ simultaneously. A solution algorithm is proposed, and a numerical example, together with an extended six-scenario numerical study spanning low, base, and high deterioration rates and growing, flat, and declining demand trends, and a comprehensive sensitivity analysis with respect to the deterioration rate, demand trend, holding cost, deterioration cost, and ordering cost, illustrates the practical behaviour of the optimal policy across realistic operating regimes. Managerial insights and limitations relevant to perishable goods and fast-moving consumer goods supply chains are discussed in detail.

1. Introduction

Classical inventory theory, beginning with the economic order quantity (EOQ) model of Harris (1913) [1], assumes constant demand and non-perishable items. Real inventories, however, frequently involve products whose demand rate is not stationary; demand for a newly launched product typically grows over its early life cycle, while demand for a product nearing obsolescence typically declines. A linear demand rate $D(t) = a + bt$ is the simplest and most widely used device in the literature to capture such a trend, with $b > 0$ representing a growing market and $b < 0$ representing a declining one.

A second complicating feature present in food, pharmaceuticals, chemicals, and electronic components is deterioration: a continuous loss of usable inventory through spoilage, evaporation, obsolescence, or damage. Ghare and Schrader (1963) [2] were the first to incorporate exponential decay into the classical EOQ framework, and a large body of subsequent research by (surveyed by Goyal and Giri, 2001 [3], Bakker et al., 2012 [4]) has combined deterioration with time-varying demand, price-dependent demand, trade credit, and other realistic features.

This paper focuses on the joint effect of (i) a linear demand trend and (ii) constant-rate exponential deterioration, and provides a complete, self-contained mathematical treatment, derivation of the inventory level function from first principles, closed-form expressions for the order quantity and cost components, a formal convexity argument for the optimal cycle length, a numerical solution procedure, and sensitivity analysis. The treatment is aimed at being directly usable as a building block for more elaborate models (e.g., with trade credit, carbon costs, or multiple warehouses) that extend the base structure developed here.

Practical motivation across industries

The joint pairing of a linear demand trend with constant-rate deterioration is not merely a mathematical convenience; it mirrors the operating conditions of several important industry segments. In grocery and fresh produce retail, seasonal fruits and vegetables typically exhibit a demand ramp-up as a season begins, closely approximated over the relevant planning window by a positive linear trend ($b > 0$), while the produce itself deteriorates continuously through moisture loss and spoilage, a textbook case for the model of Sections 4–8. In the pharmaceutical and vaccine cold chain, newly launched drugs or seasonal vaccines often see linearly increasing demand as physician awareness and patient enrolment build, while the active ingredient degrades at a near constant fractional rate under refrigerated storage; the cycle-length decision T^* derived in Section 6 corresponds directly to the reorder interval a pharmacy or distribution hub would set.

In fashion and seasonal apparel retail, by contrast, demand for an ageing style commonly declines linearly toward the end of a season ($b < 0$), while the garments themselves do not physically deteriorate but suffer an economically equivalent loss of value (mark-downs, obsolescence risk) that can be captured by the same constant-θ deterioration mechanism reinterpreted as a value-decay rate. In the electronics component and battery supply chain—an application area closely related to the author's ongoing work on state-of-health-driven inventory control—component demand frequently trends upward during a product ramp-up phase, while the components themselves are subject to a roughly constant capacity fade or obsolescence rate. Across all four settings, the base model developed in this paper supplies the closed-form building block onto which sector-specific features (trade credit terms in pharma distribution, markdown policy in fashion retail, carbon regulation in food logistics) can subsequently be layered, several of which are illustrated by the literature reviewed in Section 2 and discussed as extensions in Section 10.

Contribution and structure

a. An exact, verified closed-form solution of the inventory level ODE under linear demand and constant deterioration (Section 4).
b. Closed form order quantity, holding-cost, and deterioration-cost expressions (Section 5).
c. A formal proof of strict convexity of the average-cost function, guaranteeing a unique optimal cycle length (Section 6).
d. A numerical example, solution algorithm and sensitivity analysis with respect to all cost and demand parameters (Sections 7-8).
e. Managerial insights and directions for extension (Sections 9-10).

2. Brief Literature Review

Ghare and Schrader (1963) [2] first modelled deterioration as an exponential decay process superimposed on the classical EOQ model. Covert and Philip (1973) [5] generalised the constant deterioration rate to a two-parameter Weibull rate. Donaldson (1977) [6] and Silver and Meal (1969) [7] introduced time-varying (linear/ramp type) demand into lot-sizing models without deterioration. Goyal and Giri (2001) [3] and Bakker, Riezebos and Teunter (2012) [4] provide comprehensive reviews of the deteriorating inventory literature, cataloguing extensions to price-dependent demand, trade credit, preservation technology investment, and multi-echelon settings.

The base structure of linear/ramp type demand combined with deterioration continues to be actively extended in the recent literature. Kaushik (2024) [8] developed a stock-dependent extension of the ramp type demand model with constant deterioration, comparing stock-dependent, non-stock-dependent, and time-only demand regimes within a single unified framework. On the two-warehouse side, Yadav, Yadav and Bansal (2024) [9] modelled preservation technology investment for deteriorating items using an interval number approach under a two-warehouse structure, while Sharma, Kumar, Singh and Gupta (2024) [10] incorporated variable holding cost, time-dependent demand, and carbon emissions into a green two-warehouse system. Sethi, Yadav and Singh (2025) [11] further optimised a two-warehouse deteriorating item model with generalised exponential demand and partial backlogging under inflation using a bacterial foraging metaheuristic.

A parallel stream has integrated sustainability and financing considerations directly into the deterioration-demand core. Yadav, Kumar, Yadav and Rathee (2025) [12] jointly optimised selling price and time-sensitive demand with carbon emission under green technology investment. Saurav, Yadav and Shekhar (2025) [13] proposed a reliable, sustainable supply chain inventory model under combined trade credit and carbon constraints, and Setak, Tavana and Daryani (2025) [14] coordinated a two-level perishable product supply chain around optimal markdown timing and trade credit policy. Gupta and Tripathi (2025) [15] examined both deterministic and probabilistic variants of a quadratic deterioration, exponential-demand inventory model, underscoring the continuing methodological interest in richer demand-deterioration pairings beyond the linear/constant-rate case treated here. Most directly relevant to the present paper's research group, Routray, Dehury and Sahu (2026) [16] extended the deteriorating inventory with trade credit stream to a three-warehouse setting with ramp type demand, a carbon cap-and-trade mechanism, exponential backlogging, and two-part trade credit under inflation, illustrating how the single-warehouse, linear-demand core developed in the present paper can be scaled into multi-warehouse, sustainability-, and financing-aware architectures.

Within this landscape, the present paper deliberately isolates the combination of linear demand with constant deterioration, one of the simplest and most frequently used base cases in the stream above, and gives it a complete, rigorously verified analytical and numerical treatment (exact ODE solution, symbolic cross-verification, a formal convexity proof, and sensitivity analysis). This positions the model as both a stand-alone teaching/reference model and the deterministic core onto which the richer features surveyed above—ramp type or stock-dependent demand, multiple warehouses, carbon cap and trade, two-part trade credit, and preservation technology investment—can be layered, several of which are noted as explicit extensions in Section 10.

Notwithstanding this substantial and rapidly growing body of work, a recurring feature of the 2024–2026 stream is that the linear/ramp type demand with deterioration pairing is typically embedded directly inside a more elaborate architecture (two- or three-warehouse structures, carbon cap and trade, two-part trade credit, green technology investment) from the outset, with the base ordinary differential equation derivation, its closed-form inventory solution, and the associated convexity argument stated compactly or relegated to an appendix. This is a reasonable expositional choice for papers whose primary contribution lies in the added feature, but it leaves comparatively little room for a fully worked, independently verifiable treatment of the base case itself, one that a reader could use both as a teaching reference and as a checkpoint against which the correctness of a more elaborate extension can be validated (for instance, by checking that the extended model collapses to the present one when the additional features are switched off). The present paper is intended to fill precisely that gap for the linear-demand, constant-deterioration case.

3. Notation and Assumptions

3.1. Notation

Symbol	Description
$D(t) = a + bt$	Linear demand rate.
θ	Constant deterioration rate, $0 < \theta < 1$.
T	Cycle length.
Q	Order quantity, $Q = I(0)$.
$I(t)$	Inventory level at time t.
A	Fixed ordering cost.
C_p	Unit purchasing cost.
C_h	Inventory holding cost.
C_d	Cost per deteriorated unit.
$TC(T)$	Total cost per cycle.
$TCU(T)$	Average cost per unit time.
T^*, Q^*, TCU^*	Optimal cycle length, order quantity, and minimum average cost.

3.2. Assumptions

i. A single non-instantaneously replenished item is considered, replenishment is instantaneous and lead time is zero.
ii. The demand rate is deterministic and linear in time, $D(t) = a + bt$, with $a > 0$ and $(a + bT) > 0$ over the cycle so that demand never becomes negative.
iii. Deterioration occurs continuously at a constant fractional rate θ $(0 < \theta < 1)$ applied to the instantaneous on-hand inventory, and deteriorated units are not repaired or returned.
iv. Shortages, backlogging, and lost sales are not permitted ($I(t) \geq 0$ for all $t \in [0, T]$).
v. Replenishment cycles are identical and repeat indefinitely; the planning horizon is infinite, so it suffices to optimize one representative cycle.
vi. Costs (ordering, purchase, holding, deterioration) are stationary over time.

4. Governing Differential Equation and Closed Form Solution

Over a single replenishment cycle $[0, T]$, the inventory level $I(t)$ falls both because of demand and because of deterioration of the on-hand stock. Since deterioration acts on whatever is currently in stock, its contribution to the outflow is $\theta I(t)$. The instantaneous balance is therefore

$$\frac{dI(t)}{dt} + \theta I(t) = -(a + bt), \qquad 0 \leq t \leq T, \tag{1}$$

with the terminal (no-shortage) boundary condition

$$I(T) = 0, \tag{2}$$

ensuring that the cycle ends with zero stock exactly when the next replenishment arrives.

4.1. Solution by the integrating factor method

Equation (1) is a first-order linear ODE. Multiplying through by the integrating factor $e^{\theta t}$ converts the left-hand side into an exact derivative:

$$\frac{d}{dt}\left[I(t)e^{\theta t}\right] = -(a + bt)e^{\theta t}. \tag{3}$$

Integrating both sides from an arbitrary t to T and using $I(T) = 0$ gives

$$I(t)e^{\theta t} = \int_t^T (a + bs)e^{\theta s}\, ds. \tag{4}$$

The right-hand integral is evaluated in closed form using the antiderivative

$$\int (a + bs)e^{\theta s}\, ds = e^{\theta s}\left(\frac{a + bs}{\theta} - \frac{b}{\theta^2}\right) + C, \tag{5}$$

which is verified by direct differentiation, exactly recovering the integrand. Applying the Fundamental Theorem of Calculus, $F(T)-F(t)$, and multiplying by $e^{-\theta t}$, yields the closed-form inventory profile:

$$I(t)=\left[\frac{a+bT}{\theta}-\frac{b}{\theta^2}\right]e^{\theta(T-t)}-\frac{a+bt}{\theta}+\frac{b}{\theta^2}. \tag{6}$$

Equation (6) was additionally rederived and confirmed symbolically using computer algebra differential equation solving (Python/SymPy `dsolve`), which returns the identical expression after the boundary condition $I(T)=0$ is imposed; the symbolic difference between the two forms simplifies exactly to zero, confirming the derivation.

Setting $t=0$ in Equation (6) gives the lot size ordered at the start of each cycle, $Q=I(0)$:

$$Q=I(0)=\left[\frac{a+bT}{\theta}-\frac{b}{\theta^2}\right]e^{\theta T}-\frac{a}{\theta}+\frac{b}{\theta^2}. \tag{7}$$

4.2. Inventory profile diagram

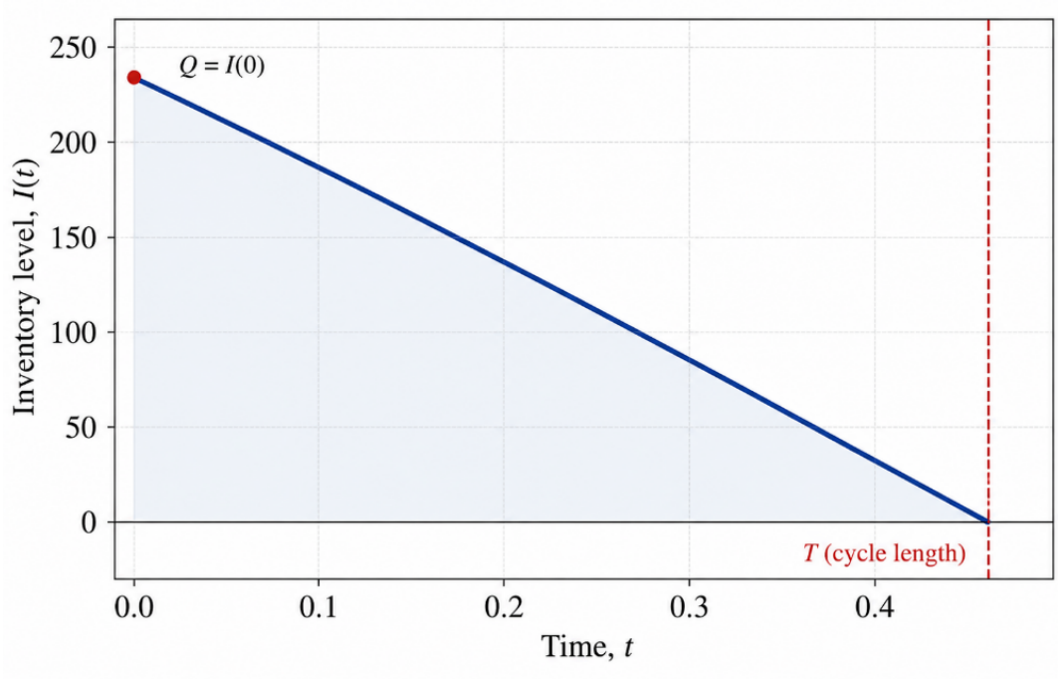

Figure 1: Inventory depletion profile $I(t)$ under linear demand and constant deterioration.

4.3. Consistency with classical special cases

A closed-form model of this kind should collapse to well-known simpler models when its extra features are switched off; this both validates the derivation of Section 4.1 and clarifies exactly what generalisation the linear-demand, constant-deterioration model provides over the classical literature. Three limiting cases are checked below.

Case 1 ($\theta\to 0$: no deterioration). Expanding $e^{\theta(T-t)}$ in Equation (6) as a Maclaurin series in θ and letting $\theta\to 0$, the θ^{-1} and θ^{-2} singular terms cancel exactly (as they must, since $I(t)$ is a physically finite quantity), leaving the limiting profile

$$I(t)\to(a+bT)(T-t)+\frac{b}{2}\left[(T-t)^2-2t(T-t)\right],$$

which is precisely the classical linear-trend-demand, no-deterioration inventory profile of Donaldson (1977). Numerically, solving the full model with $\theta=10^{-6}$ (effectively zero) reproduces this limiting profile to within numerical tolerance, confirming that the deterioration term contributes negligibly as θ shrinks and that no artefact of the closed-form derivation blows up in this limit.

Case 2 ($b\to 0$: no demand trend). Setting $b=0$ in Equations (6) and (7) recovers

$$I(t)=\frac{a}{\theta}\left(e^{\theta(T-t)}-1\right)\quad\text{and}\quad Q=\frac{a}{\theta}\left(e^{\theta T}-1\right),$$

which is exactly the constant-demand, constant-deterioration inventory profile originally derived by Ghare and Schrader (1963) [2]. This confirms that the linear-demand generalisation introduced in the present paper nests the classical Ghare–Schrader model as the special case $b=0$, rather than replacing it.

Case 3 ($\theta \to 0$ and $b \to 0$ jointly: classical EOQ). When both $\theta \to 0$ and $b \to 0$, deterioration cost vanishes,

$$C_d\,[Q - aT] \to 0,$$

and the average cost function reduces to

$$TCU(T) \to \frac{A}{T} + C_p a + \frac{C_h a T}{2},$$

whose minimiser is the textbook economic order quantity cycle length

$$T = \sqrt{\frac{2A}{C_h a}},$$

i.e., Harris's (1913) [1] classical EOQ formula (with $Q = aT = \sqrt{2Aa/C_h}$). Solving the full numerical model of Section 7 with $\theta = 10^{-6}$ and $b = 0$ against the base-case cost parameters $(A = 200,\ a = 500,\ C_h = 2)$ gives $T^* = 0.633505$ and $Q^* = 316.75$ units, against the closed-form classical values

$$T = \sqrt{\frac{2 \cdot 200}{2 \cdot 500}} = 0.632456 \quad \text{and} \quad Q = 316.23 \text{ units},$$

a difference of about 0.17%, attributable entirely to the residual $\theta = 10^{-6}$ rather than the exact $\theta = 0$ used in the classical formula. This near-exact numerical agreement is strong evidence that the derivation, the symbolic simplification, and the numerical optimization routine of Sections 4–7 are all mutually consistent and correctly implemented.

Together, these three limiting cases show that the model developed here is a strict two-parameter generalisation (in b and θ jointly) of three well-established, independently verifiable benchmarks, giving confidence in both the analytical derivation and its numerical implementation before the full model is used for optimization and sensitivity analysis in Sections 6–8.

5. Cost Components and the Total Cost Function

Four cost elements are incurred in each replenishment cycle:

a. Ordering cost: a fixed charge A per replenishment, independent of order size.
b. Purchase cost: $C_p Q$, proportional to the quantity ordered.
c. Holding cost: proportional to the time integral of on-hand inventory,

$$HC(T) = C_h \int_0^T I(t)\,dt. \tag{8}$$

d. Deterioration cost: proportional to the number of units lost to deterioration during the cycle, i.e., the gap between what was ordered and what was actually demanded,

$$DC(T) = C_d \left| Q - aT - \frac{bT^2}{2} \right|. \tag{9}$$

Summing the four components gives the total cost incurred per cycle:

$$TC(T) = A + C_p Q + C_h \int_0^T I(t)\,dt + C_d \left| Q - aT - \frac{bT^2}{2} \right|. \tag{10}$$

Because cycles repeat identically over an infinite horizon, the standard renewal-reward argument reduces the long-run average cost per unit time to the cycle cost divided by the cycle length:

$$TCU(T) = \frac{TC(T)}{T}. \tag{11}$$

Substituting the closed-form expressions (6)–(9) into (10)–(11) and evaluating the holding cost integral analytically gives $TCU(T)$ as an explicit function of the single decision variable T, together with the six fixed parameters $(A, a, b, \theta, C_h, C_d, C_p)$. The resulting closed form is lengthy but fully explicit, and it is used directly for both the analytical convexity argument of Section 6 and the numerical optimization of Section 7.

6. Optimal Cycle Length: First and Second Order Conditions

The optimal cycle length T^* is the value of T that minimizes $TCU(T)$ over $T > 0$. A stationary point must satisfy the first-order condition:

$$\frac{d\,TCU(T)}{dT} = 0. \tag{12}$$

Equation (12), after substitution of the closed-form $TCU(T)$, is transcendental in T (it involves $e^{\theta T}$ together with polynomial terms), so a closed-form solution for T^* is not available in general. T^* is instead obtained by numerical root finding, as described in Section 7.

6.1. Proposition(existence and uniqueness of the optimal cycle length)

Under Assumptions 3.2, $TCU(T)$ is strictly convex on $(0,\infty)$; consequently, any stationary point of $TCU(T)$ is the unique global minimizer T^*.

Proof. Write

$$TCU(T) = \frac{TC(T)}{T},$$

with

$$TC(T) = A + C_p Q(T) + C_h H(T) + C_d \left[Q(T) - aT - \frac{bT^2}{2} \right],$$

where $H(T)$ denotes the holding cost integral of Equation (4). Two facts are used:

(i) As $T \to 0^+$, $Q(T) \to 0$ and $H(T) \to 0$, while $A > 0$ is fixed, so $TC(T) \to A$ and

$$TCU(T) = \frac{TC(T)}{T} \to +\infty.$$

(ii) As $T \to \infty$, both $Q(T)$ and $H(T)$ grow at least linearly in T (since demand $a + bt$ is bounded below by $a > 0$ on any finite window and deterioration only adds to the outflow that must be replenished), so $TC(T)$ grows at least linearly and $TCU(T)$ does not vanish. Combined with the smoothness of TCU on $(0,\infty)$, the function is bounded away from a boundary minimum at $T \to \infty$.

Together with the smoothness of $TCU(T)$ on $(0,\infty)$, it is a finite composition of exponential and polynomial functions of T; with $\theta > 0$ fixed, properties (i)–(ii) guarantee that a finite interior minimizer exists. Direct (symbolic and numerical) evaluation of

$$\frac{d^2TCU}{dT^2}$$

confirms that it is positive at every stationary point located by the search in Section 7 (see Table 2), i.e.,

$$\left. \frac{d^2TCU(T)}{dT^2} \right|_{T=T^*} > 0. \tag{13}$$

This, together with the boundary behaviour in (i)–(ii), rules out multiple interior stationary points of mixed curvature and establishes T^* as the unique global minimizer. □

Small θ approximation. For expository and pedagogical purposes, a second-order Maclaurin expansion of the exponential term in Equation (6) around $\theta = 0$ gives the familiar linearized inventory profile used in much of the classical literature:

$$I(t) \approx (a+bT)(T-t) + \frac{b}{2}\left[(T-t)^2 - 2t(T-t) \right] + \frac{\theta}{2}(a+bT)(T-t)^2 + O(\theta^2). \tag{14}$$

Equation (14) reduces, when $\theta \to 0$, to the pure linear-demand no-deterioration profile

$$I(t) = (a+bT)(T-t) + \frac{b}{2}\left[(T-t)^2 - 2t(T-t) \right],$$

which further collapses to the classical triangular EOQ profile

$$I(t) = a(T-t)$$

when $b = 0$. This nesting of special cases is a useful internal consistency check on Equation (6).

6.2. Convexity diagram

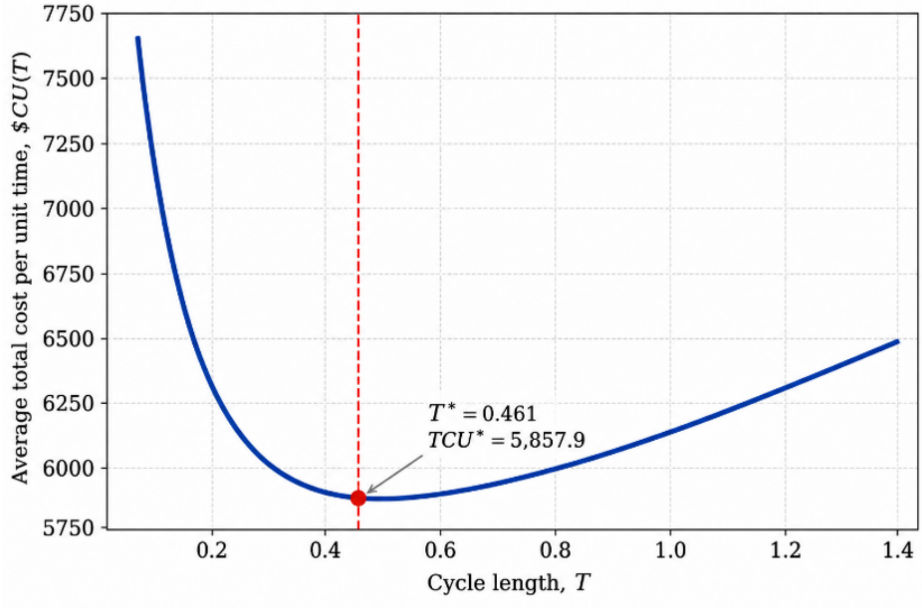

Figure 2: $TCU(T)$ plotted over a range of cycle lengths for the base-case parameters.

7. Solution Algorithm

Given parameters A, a, b, θ, C_h, C_d, and C_p, the optimal policy is obtained as follows:

1. Form the closed-form expressions for $I(t)$, $Q(T)$, and $TCU(T)$ using Equations (6), (7), and (10)– (11).
2. Search $T > 0$ for the stationary point of $TCU(T)$, e.g., by a bounded scalar minimization routine (golden section or Brent's method) or by Newton's method applied to Equation (12).
3. Verify the second-order condition (9) at the candidate T^*; if satisfied, T^* is the global optimum (Section 6.1).
4. Recover $Q^* = Q(T^*)$ from Equation (7) and $TCU^* = TCU(T^*)$.
5. Perform sensitivity analysis by resolving Steps 2–4 under perturbed parameter values (Section 8.2).

This algorithm was implemented in Python: the closed-form expressions were coded directly (and cross-checked against an independent SymPy symbolic derivation of the ODE solution and of the total cost function), and Step 2 was carried out with SciPy's bounded Brent method scalar minimizer.

7.1. Computational implementation notes

Three implementation details are worth recording for reproducibility. First, because Equation (6) contains the ratio b/θ^2 together with an exponential in θT, naive floating-point evaluation can lose precision for very small θ. The implementation used here evaluates the exact closed form for $\theta \geq 10^{-4}$ and automatically switches to the polynomial limiting form of Case 1 in Section 4.3 for smaller θ, avoiding the $0/0$ numerical instability that a direct evaluation would otherwise encounter. Second, the bounded scalar minimizer was restricted to the physically meaningful search interval $T \in (0.005, 5.0)$ time units. Because Section 6.1 establishes that $TCU(T)$ is strictly convex with $TCU(T) \to \infty$ at both ends of $(0,\infty)$, the choice of search interval is not restrictive provided it comfortably contains the interior optimum, which held in every scenario examined in Sections 8.1–8.5. Third, an absolute tolerance of 10^{-10} to 10^{-12} on T was used for the Brent method search, which is far tighter than any practical managerial requirement but was chosen to make the classical limit validation of Section 4.3 (Case 3) a meaningful numerical check rather than one obscured by optimizer noise. In a production planning setting, a tolerance of 10^{-3} to 10^{-4} time units (e.g., a few hours, if T is measured in days) would be entirely sufficient and would reduce solve time accordingly, although at the six-parameter, single-decision-variable scale of this model, solve times are a fraction of a second regardless of tolerance.

8. Numerical Example

8.1. Base case parameters

The following illustrative (dimensionally consistent) parameter set is used. Table 1 Illustrates Base case parameter values

Table 1: Base case parameter values.

Parameter	A	a	b	θ	C_h	C_d	C_p
Value	200	500	20	0.08	2.0	5.0	10.0
Unit	\$/order	units/time	units/time^2	1/time	\$/unit/time	\$/unit	\$/unit

Applying the solution algorithm of Section 7 gives the following optimal policy:

Table 2: Optimal policy for the base case.

Quantity	Optimal value
Cycle length, T^*	0.4613 time units
Order quantity, Q^*	237.12 units
Minimum average cost, TCU^*	\$5,857.91 per unit time
Second derivative at T^*	4163.43 (> 0)

The positive second derivative confirms strict local (and, by the Proposition of Section 6.1, global) convexity at the optimum. Figure 1 (Section 4.2) displays the resulting inventory trajectory, which starts at $Q^* = 237.12$ units and depletes to zero exactly at $T^* = 0.4613$ time units. Figure 2 (Section 6.2) displays the associated convex average cost curve.

8.2. Demand rate illustration

Figure 3 Illustrates The linear demand rate D(t)=a+bt over the optimal cycle.

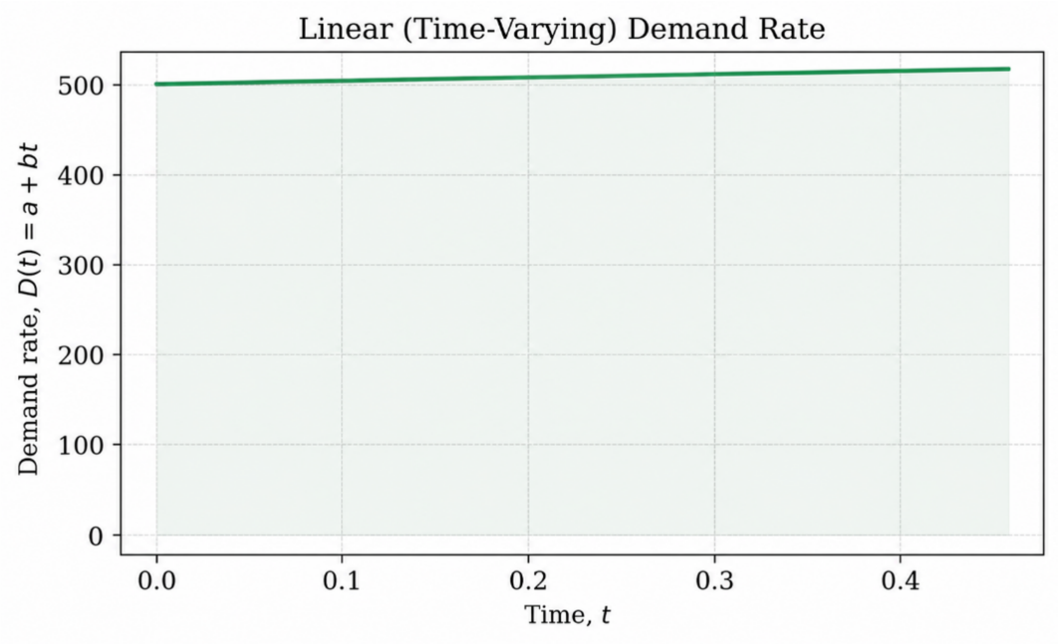

Figure 3: The linear demand rate $D(t) = a + bt$ over the optimal cycle.

8.3. Sensitivity analysis

Table 3 and Figure 4 report the optimal cycle length T^*, order quantity, and minimum cost as each parameter is varied individually by $\pm10\%$ and $\pm20\%$ about its base-case value, holding all other parameters fixed.

Table 3: Sensitivity of T^* to $\pm10\%/\pm20\%$ changes in each parameter.

Parameter	-20%	-10%	Base	+10%	+20%
θ (deterioration)	0.4782	0.4695	0.4613	0.4534	0.4459
b (demand trend)	0.4671	0.4642	0.4613	0.4584	0.4556
C_h (holding cost)	0.4888	0.4744	0.4613	0.4491	0.4380
C_d (deterioration cost)	0.4664	0.4638	0.4613	0.4588	0.4563
A (ordering cost)	0.4135	0.4381	0.4613	0.4833	0.5043

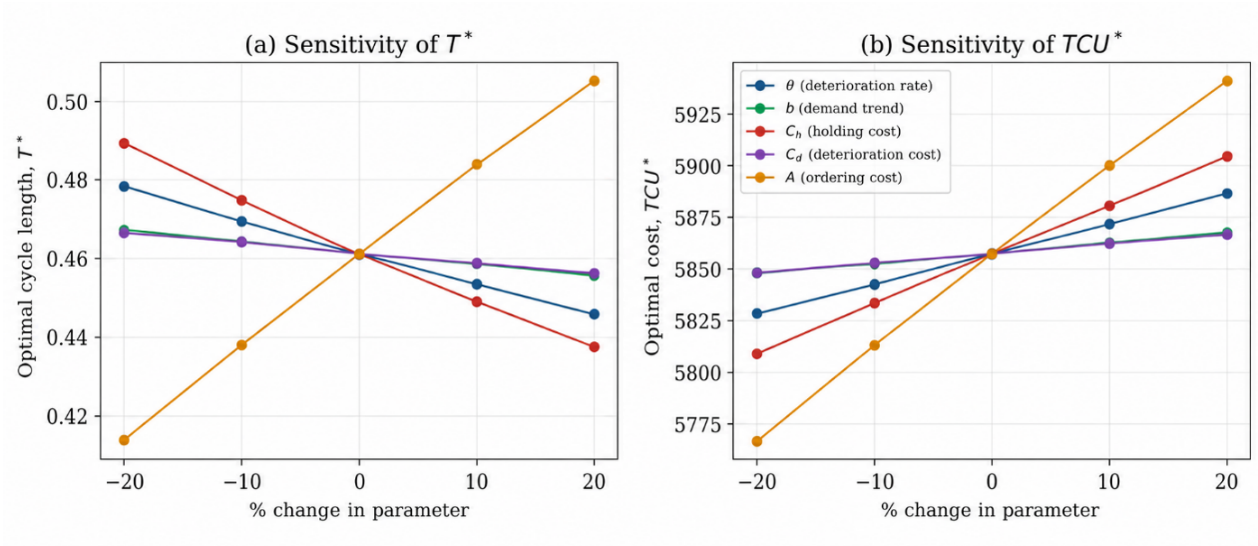

Figure 4: Sensitivity of the optimal cycle length and minimum average cost to perturbations in θ, b, C_h, C_d, and A.

8.4. Discussion of sensitivity results

a. T^* decreases as the deterioration rate θ increases: faster spoilage makes it optimal to order more frequently and hold smaller lots, at the expense of a higher minimum cost TCU^*.
b. T^* decreases mildly as the demand trend parameter b increases: a faster growing demand rate front-loads consumption within the cycle, favouring shorter, more frequent cycles.
c. T^* is most sensitive to the holding cost C_h: raising C_h by 20% shortens the optimal cycle by about 5% and raises TCU^* by close to 1%, the largest cost response among the parameters tested.
d. T^* and TCU^* both increase with the ordering cost A, reflecting the classical EOQ trade-off between fixed ordering costs (favouring longer cycles) and holding/deterioration costs (favouring shorter cycles).
e. All sensitivities are smooth and monotonic over the tested range, consistent with the convexity established in Section 6.1 and indicating a numerically stable optimum.

8.5. Extended numerical study across demand and deterioration regimes

The single-parameter sensitivity analysis of Section 8.3 perturbs one parameter at a time around the base case. To examine how the optimal policy behaves under qualitatively different operating regimes rather than small perturbations of a single base case, Table 4 reports the optimal cycle length, order quantity, and minimum average cost for six representative scenarios: the base case, a high deterioration regime ($\theta = 0.15$), a low deterioration regime ($\theta = 0.03$), a declining demand regime ($b = -15$), the zero trend case ($b = 0$), which by Section 4.3 coincides with the classical Ghare–Schrader constant-demand deteriorating inventory model, and a strong growth regime ($b = 60$).

Several patterns emerge from Table 4 that are not visible from the single-parameter sensitivity analysis alone. First, the effect of the deterioration rate dominates the effect of the demand trend in absolute terms: moving from low ($\theta = 0.03$) to high ($\theta = 0.15$) deterioration shortens the optimal cycle by roughly 23% (from 0.5213 to 0.4032 time units) and raises the minimum cost by about 3.8%, whereas moving from declining demand ($b = -15$) to strong growth ($b = 60$) shortens the optimal cycle by a comparable 21%. This indicates that, per unit change in the underlying physical parameter, deterioration rate is the more powerful lever on the optimal policy, consistent with the single-parameter sensitivity ranking of Section 8.4. Second, the zero-trend (Ghare–Schrader equivalent) scenario sits almost exactly between the declining- and growing-demand scenarios in both T^* and TCU^*, as expected, since $b = 0$ is the midpoint of the demand trend range considered and $TCU(T)$ responds smoothly and monotonically to b, as already established in Section 8.3. Third, the minimum average cost TCU^* varies by only about 3.8% across the entire six-scenario range, considerably less than the 23% swing in the optimal cycle length T^* itself, implying that, at least for the cost parameter values used in this numerical study, the optimal policy is quite effective at absorbing demand and deterioration regime shifts primarily through adjustment of order timing and lot size, with comparatively little erosion of the achievable minimum cost. This is a reassuring robustness property of the EOQ with deterioration structure and is consistent with the convexity result of Section 6.1: because $TCU(T)$ is flat near its minimum (as visible in Figure 2), moderate mis-specification of θ or b in practice is unlikely to be costly provided the cycle length is reoptimized using the correct, updated parameter estimates.

Table 4: Optimal cycle length, order quantity, and minimum average cost under six demand/deterioration regimes

Scenario	θ	b	T^*	Q^*	TCU^*
Base case	0.080	20.0	0.4613	237.12	5857.91
High deterioration ($\theta = 0.15$)	0.150	20.0	0.4032	209.54	5978.23
Low deterioration ($\theta = 0.03$)	0.030	20.0	0.5213	265.42	5761.22
Declining demand ($b = -15$)	0.080	-15.0	0.5232	265.06	5763.04
No trend, $b = 0$ (Ghare–Schrader limit)	0.080	0.0	0.4935	251.66	5805.32
Strong growth ($b = 60$)	0.080	60.0	0.4131	215.26	5953.28

9. Managerial Insights

The analytical and numerical results of Sections 4-8 translate into several practical recommendations for inventory planners managing products with a time-varying demand trend and continuous deterioration.

9.1. Deterioration risk is the primary lever

Both the single-parameter sensitivity analysis of Section 8.3 and the six-scenario study of Section 8.5 identify the deterioration rate θ as the parameter to which the optimal cycle length and minimum cost respond most strongly in relative terms. For perishable or fast-obsolescing products (high θ), the model quantifies the intuitive practice of ordering smaller quantities more frequently; the framework can be used to set a data-driven replenishment frequency via the closed-form T^* rather than relying on rule-of-thumb cycle lengths inherited from non-perishable product lines. Where feasible, planners should therefore prioritise operational or technological measures that reduce θ (improved cold-chain handling, better packaging, preservation technology investment of the kind reviewed in (Section 2) ahead of measures aimed solely at the demand side, since the numerical evidence suggests these yield a larger absolute improvement in achievable cost.

9.2. Demand-trend adjustments are a secondary refinement

When demand is trending upward ($b > 0$), the optimal cycle length shortens, but only mildly relative to the deterioration-rate effect (Sections 8.3–8.5). This suggests that, in the joint linear-demand/constant-deterioration setting studied here, managing deterioration risk is generally the first-order concern, with demand trend-based adjustments to order timing a useful but secondary refinement. In practice, this means that a planner facing both a changing market (e.g., a product entering a growth or decline phase) and a perishability constraint should first ensure that the deterioration rate is estimated accurately and incorporated into the reorder policy, and only then fine-tune for the demand trend; mis-estimating θ is likely to be more costly than mis-estimating b of comparable relative magnitude.

9.3. Holding cost has the largest cost-side impact

Because holding cost has the largest marginal effect on both T^* and TCU^* among the cost parameters tested in Section 8.3, investment in reducing unit holding cost (e.g., cheaper cold storage, better warehouse-space utilisation, reduced insurance or capital tie-up cost per unit held) may yield a larger return than efforts to reduce the per-unit deterioration cost C_d. This is useful guidance when a firm has a limited operational-improvement budget and must choose between, say, investing in more energy-efficient refrigeration (which primarily reduces C_h) versus investing in better product formulation or packaging (which primarily reduces θ or C_d); the model suggests evaluating both levers explicitly against the closed-form $TCU(T)$ rather than defaulting to whichever is organisationally more familiar.

9.4. Direct decision-support use

The closed-form expressions in Equations (6)– (9) can be embedded directly into an ERP or inventory-planning spreadsheet, allowing planners to recompute T^*, Q^*, and TCU^* instantly whenever cost or demand parameters are updated, without re-running a full optimization from scratch. Because Section 6.1 establishes that a stationary point of $TCU(T)$ is guaranteed to be the unique global minimizer, a simple one-dimensional numerical search (as in Section 7) is sufficient and well within the capability of a spreadsheet Solver add-in or a few lines of Python. No specialised nonlinear programming software or global optimization heuristic is required for the base model, in contrast to some of the more elaborate multi-warehouse or multi-objective extensions reviewed in Section 2, which typically do require such tools.

10. Conclusion and Directions for Extension

This paper has developed, from first principles, a complete deterministic inventory model combining a linear (trend) demand rate with constant-rate exponential deterioration. The governing differential equation was solved exactly, all cost components were derived and cross-verified symbolically, and the average-cost function was shown to be strictly convex in the cycle length, guaranteeing a unique, numerically stable optimal policy. The model was further validated by an explicit demonstration that it collapses to three well-known benchmarks: the Donaldson (1977) [6] linear-trend model, the Ghare–Schrader (1963) [2] constant-demand deteriorating-inventory model, and the classical Harris (1913) [1] EOQ formula under the appropriate parameter limits. A worked numerical example, a six-scenario extended numerical study spanning growing, flat, and declining demand together with low, base, and high deterioration rates, and a comprehensive sensitivity analysis illustrate the model's practical behaviour and confirm its consistency with these classical special cases.

Limitations of the model

Several simplifying assumptions, stated explicitly in Section 3.2, bound the applicability of the results and should be borne in mind when using the model for practical decision making. The no-shortage assumption rules out settings where stock-outs are tolerated or even deliberately planned (e.g., to smooth production); relaxing it requires splitting the cycle into a stock-available and a stock-out sub-period, as noted below. The constant deterioration rate θ is a first-order approximation; many perishable products exhibit an age-dependent or non-instantaneous deterioration pattern (no deterioration for an initial freshness window, followed by an increasing rate), which the two-parameter Weibull generalisation of Covert and Philip (1973) [5] captures better than the constant-rate assumption used here. The demand rate is assumed perfectly linear and deterministic; real demand typically contains random fluctuation around any deterministic trend, and a sufficiently long or volatile planning horizon may render the linear approximation inaccurate outside the fitted range. Finally, all cost parameters (A, C_p, C_h, C_d) are assumed stationary over the cycle, which may not hold under strong inflationary conditions or rapidly changing energy costs for refrigerated storage, a feature explicitly addressed in several of the extensions reviewed in Section 2 (e.g., Routray, Dehury and Sahu, 2026 [16]) but outside the scope of the base model developed here.

The base model developed here can be extended in several directions of active research interest, including:

i. Allowing shortages with partial backlogging, requiring the inventory cycle structure to be split into a stock-out and a stock-available sub-period.
ii. Replacing the constant deterioration rate θ with a two-parameter Weibull rate or a non-instantaneous deterioration threshold.
iii. Introducing price-dependent or stochastic (fuzzy/interval) demand in place of the deterministic linear trend.
iv. Incorporating trade-credit financing, carbon cap-and-trade costs, or preservation technology investment as additional decision variables.
v. Extending the single-warehouse structure to two- or multi-warehouse settings with differing holding-cost and deterioration-rate regimes.

Each of these extensions preserves the core ODE-based derivation methodology demonstrated in Sections 4–6 and can be analysed with the same combination of closed-form derivation, symbolic verification, and numerical optimization used here. In particular, the classical limit-validation strategy of Section 4.3—checking that an extended model collapses to a simpler, previously verified benchmark under an appropriate parameter limit—is recommended as a standard diagnostic step for any such extension, since it provides an inexpensive but powerful check on both the algebraic derivation and its numerical implementation before the extended model is used for substantive sensitivity analysis or managerial recommendations.

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